HLP: $p: E \to B$ has HLP with respect to $X$ if given $g_t: X \times I \to B$ and a lift $\tilde{g}_0: X \to B$, there exists a lift $\tilde{g}_t: X \times I \to B$ extending $\tilde{g}_0$.

A fibration $p: E \to B$ has HLP w.r.t all $X$.

A fiber bundle $p: E \to B$ with fiber $F$ has the prop. that $B$ is covered by open sets $U$ with $p^{-1}(U) \xrightarrow{h} \overset{\times}{\to} U \times F$.

Ex: $S' \to S^{2n+1} \to \mathbb{C}P^n$ [Gen. Hopf bundle]

\[ S^{2n+1} = \{ z \in \mathbb{C}^n \mid \sum |z_i|^2 = 1 \} \]

$\overset{p}{\rightarrow}$

$\mathbb{C}P^n = \mathbb{C}^n \setminus \{0\} / \mathbb{C}^*$

$S'$ acts freely on $S^{2n+1}$ by $\lambda \cdot z = \lambda z$

Orbits are exactly the fibers of $p$. 
Generated by some nonehren vector field, so locally fibers are a stack of spaghetti.

To see product structure, find a section $U \rightarrow S^{2n+1}$ and then use the action:

$$U = \left\{ \left[ z_0 : \ldots : z_n \right] \mid z_0 \neq 0 \right\}$$

$$s(U) = \left\{ z \in S^{2n+1} \mid z_0 \in \mathbb{R}_+ \right\}$$

$$s \left( \left[ z_0 : \ldots : z_n \right] \right) = \frac{|z_0|}{z_0 |z|} \cdot z \quad \text{where} \quad z = (z_0, \ldots, z_n)$$

$$|z| = \sqrt{\sum |z_i|^2}$$

Define $h^1: U \times S^1 \rightarrow p^{-1}(U)$ by

$$([z], \lambda) \mapsto \lambda s([z])$$

Its inverse is $h(z) = (p(z), z_0 / |z_0|)$ so we've proved the local product structure.

Thm: A fiber bundle $p: E \rightarrow B$ where $B$ is paracompact is a fibration.

---

every open cover has a locally compact refinement.
Cor: \( \pi_3 S^2 = \mathbb{Z} \), gen by Hopf map \( S^3 \to S^2 \).

Pf: The Hopf bundle \( S^1 \to S^3 \to S^2 \) gives a long exact sequence:

\[
\pi_n S^1 \to \pi_n S^3 \xrightarrow{P_*} \pi_n S^2 \to \pi_{n-1} S^1.
\]

and so \( \pi_n S^3 \cong \pi_n S^2 \) for \( n > 3 \).

Cor: \( CP^\infty \) is a \( K(\mathbb{Z}, 2) \).

Pf: Previous construction yields a fiber bundle

\[
S^1 \to S^\infty \to CP^\infty.
\]

Thm: A fiber bundle has H.L.P. with respect to all CW complexes \( X \). [For long exact seq, only really need HLP with respect to \( D^n \).]

Pf is purely local:

\[
\text{cover by finitely many product } A
\]
Bott Periodicity:

Fiber bundle: \( O(n-1) \to O(n) \to S^{n-1} \)

So \( \pi_i^* O(n) \) is at least as complicated as \( \pi_i^* (S^m) \). By long exact seq of a fibration \( O(n-1) \to O(n) \) gives an isom on \( \pi_i \) for \( i < n-2 \).

Thus \( \pi_i O(n) \) is independent of \( n \) for large \( n \), call it \( \pi_i O \). Turns out to only depend on \( i \mod 8 \):

\[
\begin{array}{ccccccccc}
 i \mod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\pi_i O & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\
\end{array}
\]

This is called Bott periodicity.

Some details: \( O(n) = \{ Ae \in GL_n \mathbb{R} \mid AA^t = I \} \)

\( O(1) \leq O(2) \leq O(3) \leq O(4) \leq \cdots \) \( (A^{0}_{0,0,0}) \)

\( \mathbb{Z}/2\mathbb{Z} \leq \mathbb{Z}/2\mathbb{Z} \times S^1 \leq RP^3 \parallel RP^3 \)

Always two comp's, can to det = \( \pm 1 \).
Theme: Infinite d-nil' spaces can have simpler $\pi_n$ than finite-d-nil' ones.

Fact: There is no simply-connected finite CW complex where all of $\pi_n$ is known, other than contractible ones.

Stable homotopy groups: Via suspension, have

$$\pi_i S^n \rightarrow \pi_{i+1} S^{n+1}$$

which is an isomorphism for $i < 2n-1$. Define

$$\pi_i^S = \pi_{i+n} S^n \text{ for } n > i+1$$

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
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<td>$\mathbb{Z}/240$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
</tr>
</tbody>
</table>

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$$\uparrow$

$id$ $\eta: S^3 \rightarrow S^2$ $S^7 \rightarrow S^4$ $S^{15} \rightarrow S^8$