Lecture 6: More on the tangent space.

Last time:

\[ T_p M = \left\{ \begin{array}{l}
 v : \mathcal{C}^\infty(M) \to \mathbb{R} \quad \forall f, g \in \mathcal{C}^\infty(M), c \in \mathbb{R} \\
 \quad \cdot \quad v(f + c g) = v(f) + c v(g) \\
 \quad \cdot \quad v(fg) = v(f) \, g(p) + f(p) \, v(g) \\
 \end{array} \right. \]

If \( f : M \to N \) is smooth, get \( dF_p : T_p M \to T_{F(p)} N \) by

\[ dF_p(v)(g \in \mathcal{C}^\infty(N)) = v(g \circ F) \]

Locality: If \( f, g \in \mathcal{C}^\infty(M) \) agree on a nbhd of \( p \), then \( v(f) = v(g) \) for all \( v \in T_p M \).

Lemma: \( U \subseteq M \) an open subset of a smooth mfd. For every \( p \in U \), the inclusion \( i : U \to M \) gives an isomorphism \( d_i_p : T_p U \to T_p M \).

Cor: \( \dim T_p M = \dim M \).
Pf of Cor: Let \((U, \varphi)\) be a smooth chart at \(p\).

Then \(\varphi: U \rightarrow \varphi(U)\) is a diffeomorphism. So

\[
\begin{align*}
T_p U & \xrightarrow{d\varphi_p} T_p \varphi(U) \\
\cong & \\
T_p M & \xrightarrow{\cong} T_p \mathbb{R}^n \leftarrow \text{has dim } n.
\end{align*}
\]

Pf of Lemma: By HW, there are open nbhds

\(W \subseteq V \subseteq U\) of \(p\) with \(\overline{V} \subseteq U\) and a smooth \(h: M \rightarrow \mathbb{R}\) where \(h = 1\) on \(W\) and \(h = 0\) outside of \(V\).

By locality, if \(f \in C^\infty(M)\) then

\(v(h \cdot f) = v(f)\) for \(v \in T_p M\). Thus \(v\in T_p M\)

is determined by its values on

\[\{ f \in C^\infty(M) \mid f \text{ vanishes outside } V_p \}\]

and any derivation on \(\uparrow\) gives an elt of \(T_p M\).
The same is true for $U$, i.e. can identify $T_p U$ with derivations on $\{f \in C^\infty(U) \mid f \text{ vanishes outside } V\}$.

Since these two sets of funs are equal (just extend any $f \in C^\infty(U)$ vanishing outside $V$ by 0 outside $U$) we get $T_p U \cong T_p M$.

Other points of view:

1. $p, U, \varrho, V = 0$

   $\nabla \in T_p M$ is $d\varrho^{-1}_{\varrho(p)} \nabla$

   **Claim**: since $d\varrho^{-1}_{\varrho(p)} : T_{\varrho(p)} \varrho(U) \to T_p M$, $1 \in T_{\varrho(p)} \mathbb{R}^n$

Local coordinates: $\mathbb{R}^n$ with coordinates $x_1, x_2, \ldots, x_n$

Standard basis for $T_p \mathbb{R}^n = \{e_1, e_2, \ldots, e_n\}$

$e_i = (0, \ldots, 1, \ldots, 0)$  \hspace{1cm} $i$th place

Since also think of $T_p \mathbb{R}^n$ as directional derivatives/derivations
often denote $e_i$ as $\frac{\partial}{\partial x_i}$. Given $(U, \varphi)$, define

$$\frac{\partial}{\partial x_i} |_p \circ (d\varphi_p)^{-1} (\frac{\partial}{\partial x_i} |_{\varphi(p)}) = d(\varphi^{-1})_p \left( \frac{\partial}{\partial x_i} |_{\varphi(p)} \right)$$

to get a basis for $T_p M$. These work like you expect:

Matrix of $dF_p$ with respect to $\frac{\partial}{\partial x_i} |_p$ and $\frac{\partial}{\partial y_i} |_{F(p)}$ is $D_{\varphi(p)}(\psi \circ F \circ \varphi^{-1})$.

2. Germs. [Likely skip, refer to text.]

$$C_p^\infty(M) = \left\{ (U, f) \mid f \in C^\infty(U), \frac{\partial}{\partial x_i} |_p \right\} / (U, f) \sim (V, g)$$

Can work with derivations on this instead.
Curves: A smooth \( \gamma: J \to M \) where \( J \subseteq \mathbb{R} \) is an interval. The velocity of \( \gamma \) at time \( t_0 \) is

\[
\gamma'(t_0) = \frac{d\gamma}{dt}(t_0) \in T_{\gamma(t_0)} M
\]

Props: [Check on your own.]

a) If \( f \in C^\infty(M) \), then \( \gamma'(t_0)(f) = (f \circ \gamma)'(t_0) \)

b) If \( F: M \to N \) is smooth, then

\[
dF_{\gamma(t_0)}(\gamma'(t_0)) = (F \circ \gamma)'(t_0)
\]

c) Every \( v \in T_p M \) is \( \gamma'(t_0) \) for some \( \gamma: J \to M \).

\[
\gamma_1 \sim \gamma_2 \text{ if } \quad \forall f \in C^\infty(M) \text{ we have } (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)
\]

\[\mathcal{V}_p M = \left\{ \text{smooth curves } \gamma: J \to \mathbb{R} \mid \gamma(0) = p \right\} \]

Problem 3-8 (on HW #3) shows \( \mathcal{V}_p M \cong T_p M \).