**Lecture 22: Covector fields**

Vector space \( V^* = \{ f : V \to \mathbb{R} \mid f \text{ linear transformation} \} \)

Cotangent bundle: \( T^*M = \bigsqcup_p T^*_p M \) where \( T^*_p M = (T_p M)^* \)

Covector field: \((p \in M) \mapsto (\omega_p \in T^*_p M)\)

\[
\omega = xy \, dx + e^{x^2 + y} \, dy
\]

[Just as for vector fields, there is a notion of]

\[
\Omega^1(M) = \left\{ \text{smooth covector fields on } M \right\}
\]

- Vector space \( \mathbb{C}^{\infty}(M) \)-module

Also called "differential 1-forms".

**Pull backs:** \( F : M \to N \) smooth. For \( \omega \in \Omega^1(N) \)

Define \( F^*(\omega) \in \Omega^1(M) \) by \( (F^*\omega)_p = (dF_p)^*(\omega_{F(p)}) \)

\[
(F^*\omega)_p (v) = (\omega_{F(p)})_{F(p)} (dF_p (v))
\]

\( M \xrightarrow{F} N \)
Unlike vector fields, where $F^* X$ only sometimes makes sense, can always pull back a vector field.

**Ex:** $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\
(x, y) \rightarrow (u, v) \\
F(x, y) = (x + y^2, x^2)$

\[
\omega = u \, du + v \, dv \quad \text{DF} = \begin{pmatrix} 1 & 2y \\ 2x & 0 \end{pmatrix}
\]

\[
(F^* \omega)_{(x, y)} \left( \frac{\partial}{\partial x} \right) = \omega_{(x + y^2, x^2)} \left( dF_{(x, y)} \left( \frac{\partial}{\partial x} \right) \right)
\]

\[= \omega_{(x + y^2, x^2)} \left( \frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v} \right)
\]

\[= x^2 + 2x
\]

\[
(F^* \omega)_{(x, y)} \left( \frac{\partial}{\partial y} \right) = \omega_{(x + y^2, x^2)} \left( 2y \frac{\partial}{\partial u} \right)
\]

\[= 2yx^2
\]

So $F^* \omega = (x^2 + 2x) \, dx + 2yx^2 \, dy$

**Check:** (Uses HW!) \[ (F^* \omega)_{(x, y)} = (dF_{(x, y)})^* (\omega_{(x + y^2, x^2)}) \]

\[= \begin{pmatrix} 1 & 2x \\ 2y & 0 \end{pmatrix} \begin{pmatrix} x^2 \\ 1 \end{pmatrix} = \begin{pmatrix} x^2 + 2x \\ 2yx^2 \end{pmatrix}
\]

transpose
For \( M \xrightarrow{f} N \xrightarrow{g} S \) have \( \Omega^1(M) \xrightarrow{f^*} \Omega^1(N) \xrightarrow{g^*} \Omega^1(S) \)

\[ f^* g^* = (g \circ f)^* \]

For \( f : M \to \mathbb{R} \), the differential \( \omega_f \in \Omega^1(M) \) is defined by \( \omega_f (v_p \in T_p M) = v_p (f) \).

[This is the "gradient-like" vector field from Wed.]

For \( G : N \to M \), we have \( \omega_{f \circ G} = G^*(\omega_f) \) since

\[
\omega_{f \circ G}(v_q) = V_q (f \circ G) = (dG_q (V_q)) f = (\omega_f)_{G(q)} (dG_q (V_q)) = (G^* \omega_f)(V_q)
\]

Thm: \( f : M \to \mathbb{R} \) smooth. Then \( \omega_f = f^*(dt) \) where \( dt \in \Omega^1(\mathbb{R}) \) is the dual to \( \frac{\partial}{\partial t} \).

Pf: Let \( i : \mathbb{R} \to \mathbb{R} \) be the identity map. Then

\[
\omega_i \left( \frac{\partial}{\partial t} \bigg|_p \right) = \left( \frac{\partial}{\partial t} \bigg|_p \right)(i) = \alpha = dt \left( \frac{\partial}{\partial t} \bigg|_p \right)
\]

and \( i^*(dt) = dt \). The result for general \( f : M \to \mathbb{R} \) now follows by \( \otimes \) since
\[ \omega_f = \omega_{i \circ f} = f^*(\omega_i) = f^*(d\tau). \]

Notation: The usual notation for \( \omega_f \) is \( df \).

Unfortunately, this is also our notation for the derivative \( \frac{df}{df} : TM \rightarrow TR \). Both \( \omega_f \) and our original \( df \) "eat" tangent vectors, but the former outputs a elt of \( TR \) and the latter an elt of \( T_{f(p)} R \). Specifically,

\[ df(v_p) = \omega_f(v_p) \frac{\partial}{\partial t} \bigg|_{f(p)} \]

or

\[
\begin{array}{c}
TM \\ \xrightarrow{df} \end{array} TR \xrightarrow{\pi} R \xrightarrow{df} \left( p, \frac{\partial}{\partial t} \bigg|_{f(p)} \right) \mapsto a
\]

From now on, will denote \( \omega_f \) by \( df \) (since this the standard notation) and continue yet to denote \( TM \rightarrow TR \) by \( df \).
Note: On $\mathbb{R}^n$ with coor $(x_1, \ldots, x_n)$ then
\[ w_{x_i} = \text{"formal } dx_i;\]
\[ \uparrow \text{ dual basis to } \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}. \]

Integration: $[a, b] \subseteq \mathbb{R}$ bounded interval

For $\omega \in \Omega^1([a, b])$ define
\[ \int_{[a, b]} \omega = \int_a^b f(t) \, dt \]
where $\omega = f(t) \, dt$

Thm: Suppose $F: [c, d] \rightarrow [a, b]$ is a

diffeomorphism. Then $\forall \omega \in \Omega^1([a, b])$ one has
\[ \int_{[a, b]} \omega = \int_{[c, d]} F^*(\omega) \]