Lecture 21: Prelude to integration

The story so far: $M^n$ smooth, $F: M \to N$ smooth

$T_p M = \{ \text{directional derivatives} \} = \{ \text{velocity vectors of} \}$, $dF: TM \to TN,$

Vector fields, $[X, Y], ...$ [All about derivatives.]

On first day, said a smooth manifold something locally like $\mathbb{R}^n$ on which we can do calculus. [Query.]

So far mostly just been differentiation, but there's one exception.

Next focus: What can we integrate on a smooth manifold? [Need some kind of additional str.]

All diffeomorphic, so "the same" to us. [Let's think back to vector calculus... ]
Curve in $\mathbb{R}^3$, $\gamma: [a, b] \rightarrow C$

Parameterization

$\text{Length}(C) = \int_a^b |\gamma'(t)| \, dt = \int_C ds$

$f: C \rightarrow \mathbb{R}$ define

$\int_C f \, ds = \int_a^b f(\gamma(t)) |\gamma'(t)| \, dt$

$\text{F vector field}$

$\int_C \text{F} \cdot ds = \int_a^b \text{F}(\gamma(t)) \cdot \gamma'(t) \, dt$

$= \int_C (\text{F} \cdot \text{T}) \, ds$

$\uparrow \quad \text{T = unit tangent vector field}$

Riemannian motion

Both generalize to surfaces, etc.

Differential forms

Our focus leads to general Stokes theorem

Query: What are some differences?

One is that $\int_C \text{F} \cdot ds$ depends on the orientation of $C$
First some algebra...

$V$ vector space [over $\mathbb{R}$]. Consider the dual $V^* = \{ \alpha : V \to \mathbb{R} \mid \alpha \text{ linear} \}$ "Space of covectors" which is also a vector space: $\alpha, \beta \in V^*$ define $\alpha + \beta \in V^*$ by $(\alpha + \beta)(v) = \alpha(v) + \beta(v)$.

Key facts: Assume $V$ is finite dimensional

1. $\dim V^* = \dim V$
   If $e_1, \ldots, e_n$ is a basis for $V$, define $w^i \in V^*$ by $w^i(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ [Good exercise to see this is a basis.]

2. If $F : V \to W$ is a linear transformation, get $F^* : W^* \to V^*$ by $F^*(\beta) = \beta \circ F$

If $G : W \to X$, then

$$(G \circ F)^* = F^* \circ G^*$$

3. There is a natural isomorphism

$$V \to (V^*)^* \text{ given by } v \mapsto \delta_v$$
where $\gamma_V^*(\beta) = \beta(V)$.

**Ex:** $e_1, e_2$ std basis for $\mathbb{R}^2 = V$
\[
\alpha = \omega^1 - 2\omega^2 \quad \omega^1, \omega^2 \text{ dual basis}
\]
\[
\ker \alpha = \langle 2e_1 + e_2 \rangle
\]

\[
f_1 = e_1 \quad f_2 = e_1 + e_2 \quad \eta^1, \eta^2 \text{ dual basis}
\]

**IMPORTANT:** $e_i \leftrightarrow \omega^i$ and $f_i \leftrightarrow \eta^i$

induce different maps $V \leftrightarrow V^*.$

\[
equal \text{vectors} \quad \begin{cases} 
  e_1 - 2e_2 & \omega^1 - 2\omega^2 \\
  3f_1 - 2f_2 & \eta^1 - \eta^2
\end{cases}
\]

\[
\alpha = a\eta^1 + b\eta^2
\]

\[
\alpha(f_1) = a \quad \alpha(f_2) = b
\]

\[
\alpha(e_1) = 1 \quad \alpha(e_1 + e_2) = -1
\]
$M$ smooth $n$-mfld. The cotangent space to $M$ at $p$ is $T^*_pM = (T_pM)^*$. The cotangent bundle $T^*M = \bigsqcup_p T^*_pM$.

Local coordinates: $p \in \mathbb{R}^n$ coor $(x_1, x_2, \ldots, x_n)$

$T_pM$ has basis $\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p$

The dual basis for $T^*_pM$ is denoted $dx_1|_p, \ldots, dx_n|_p$

Cotector fields/1-forms: $\omega: (p \in M) \mapsto (\omega_p \in T^*_pM)$

Example: $f: M \to \mathbb{R}$. Consider $\omega$ defined by

$\omega_p(V_p) = V_p f$ Normally called “$df$.”

Example: $(x \cos y) \, dx + e^{x+y} \, dy$