Lecture 16: Vector fields, integral curves, and flows.

\[ \Theta : \mathbb{R} \times M \to M \text{ smooth action} \]
\[ (t, m) \mapsto t \cdot m = \Theta_t(m) \]

\[ \Theta_t : M \to M \text{ diffeomorphisms with } \Theta_{t_1} \circ \Theta_{t_2} = \Theta_{t_1 + t_2} \]

For \( m \in M \) have a curve \( \Theta^{(m)} : \mathbb{R} \to M \)
\[ t \mapsto \Theta_t(m) \]

**Infinitesimal generator:** \( V \in \mathfrak{X}(M) \)
\[ V_m = \frac{d}{dt} \Theta^{(m)} \bigg| _{t=0} = d\Theta \left( \frac{\partial}{\partial t} \right)_{(0, m)} \]

A curve \( \gamma : I \to M \) is an integral curve for \( X \in \mathfrak{X}(M) \) if \( \gamma'(t) = X_{\gamma(t)} \) for all \( t \in I \)

[\( \gamma \) a solution to an ODE on \( M \).]

**Ex:** Smooth \( \Theta : \mathbb{R} \times M \to M \), \( V \) the infinit gen. Then each \( \Theta^{(m)} \) is an integral curve for \( V \) as follows:
Set \( m' = \Theta^{(m')}(t) \); since \( \Theta^{(m')}(s) = \Theta^{(m)}(s+t) \)
\[ \Theta^{(m')}(t) = (\Theta^{(m')})'(0) = V_{m'} = V_{\Theta^{(m)}(t)} \]

Note: R-actions are also called flows.

Does every \( X \in \mathcal{X}(M) \) come from a flow?

No: \( M = \{ x \in \mathbb{R}^2 \mid |x| < 1 \} \)
\[ X = \frac{\partial}{\partial x} \]

Integral curve containing \((0,0)\):
\[ \gamma : (-1,1) \to M \]
\[ \gamma'(t) = \frac{\partial}{\partial x} \quad \checkmark \]

Can't enlarge the domain so can't set \( \Theta^{(0,0)} = \alpha \).

Thm: \( X \in \mathcal{X}(M) \). For each \( m \in M \) there is open interval \( I(m) \) containing 0 and a curve \( \gamma : I(m) \to M \) where:

(a) \( \gamma \) is an integral curve for \( X \) with \( \gamma(0) = m \).

(b) If \( \alpha : J \to M \) is an int curve for \( X \) with \( \alpha(0) = m \)

then \( J \subseteq I(m) \) and \( \alpha = \gamma \big|_{I(m)} \).
Pf: Given \( m \in M \), the existence of an integral curve on some \((-\varepsilon, \varepsilon)\) follows from applying the existence theorem for ODE's in some chart.

Suppose \( \alpha: I \to M \) and \( \beta: I \to M \) are integral curves with \( \alpha(0) = \beta(0) = m \).

Claim: \( \alpha = \beta \) on \( I \cap J \).

[If true then we just define \( \gamma \) by]
[taking \( I(m) = U \) (domain of some \( \xi \))]

Let \( t_0 \in I \cap J \) be such that \( \alpha(t_0) = \beta(t_0) \) but \( \exists t \text{ arb. close to } t_0 \text{ with } \alpha(t) \neq \beta(t) \). In local coor near \( \alpha(t_0) \) have \( \varphi \circ \alpha \)

Contradicts uniqueness of solutions to ODE's.

\( \varphi(\alpha(t_0)) \neq \varphi(\beta(t_0)) \)
Let $X \in \mathfrak{X}(M)$. Define

$$\mathcal{D} = \{(t, m) \in \mathbb{R} \times M \mid t \in I(m)\}$$

and $\Theta : \mathcal{D} \to M$ by $(t, m) \mapsto \gamma_m(t)$, where $\gamma_m : I(m) \to M$ is the int. curve for $X$ where $\gamma_m(0) = m$.

**Thm.** $\mathcal{D}$ is open in $\mathbb{R} \times M$ and $\Theta$ is smooth.

**Reason:** Smooth dep. of solutions to ODE's on initial conditions.

**Complete vector field:** $X \in \mathfrak{X}(M)$ where $\mathcal{D} = \mathbb{R} \times M$.

[Precisely those coming from $\mathbb{R}$-actions]

**Thm.** If $M$ is compact, then any $X \in \mathfrak{X}(M)$ is complete.

**Pf.** Cover $M$ with finitely many $V_i$ where each $V_i \subseteq (U_i, \varphi_i)$ and the closure of
\( \phi_i(V_i) \) in \( \mathbb{R}^n \) is opt an \( \leq 0 \).

By ODE theory, \( \exists \varepsilon_i \) so that \( y' = y \) has sol on \((-\varepsilon_i, \varepsilon_i)\)
for all init. cond in \( \phi(V_i) \).

So back on \( M \), any integral curve can be extended by at least time \( \varepsilon = \min \varepsilon_i > 0 \). So \( M \) is complete. \( \Box \)

**Thm:** Every left-invariant vector field on a Lie gp \( G \) is complete.

**Pf:** Integral curve exists at \( e \) for some time \( \pm \varepsilon \).
By left invariance this is true at every other \( g \in G \). So the left-inv. v.f. is complete.