Lecture 31: Exterior Differentiation

$$\Omega^k(M) = \left\{ \text{smooth k-forms} \right\}_{p \mapsto \Lambda^k(T_pM)}$$

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**Goal:** Define $d: \Omega^k(M) \to \Omega^{k+1}(M)$ generalizing $C^\infty(M) \to \Omega^1(M)$. Note that $f \mapsto df$ by definition $\Lambda^0 V = \mathbb{R}$, so $C^\infty(M) = \Omega^0(M)$.

**Exterior Algebra:** $\Lambda(V) = \bigoplus_{k=0}^{\dim V} \Lambda^k V$

Operations are addition and $\wedge$ product, which make it an associative graded (anti) commutative algebra: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, $\alpha \in \Lambda^k V$, $\beta \in \Lambda^l V$

**Algebra of differential forms:** $\Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M)$
Thm: M a smooth mfd, poss. with boundary. There are unique maps \( d: \Omega^k(M) \rightarrow \Omega^{k+1}(M) \) satisfying:

1. \( d \) is linear over \( \mathbb{R} \).
2. \( \omega \in \Omega^k(M) \) and \( \eta \in \Omega^l(M) \) then
   \[
   d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta
   \]
3. \( d \circ d = 0 \)
4. For \( f \in \Omega^0(M) \), \( df \) is the usual differential of \( f \), i.e. \( df(v_p) = V_p f \).

Some observations:

1. Anything satisfying 1 and 3 must be local, i.e. \((d\eta)_p \) is determined by \( \eta|_U \) for any \( p \in U \subseteq M \).

   Reason: Take \( \psi \) with \( \text{supp} \psi \subseteq U \) and \( \psi = 1 \) on \( p \in V \subseteq U \).

   Then
   \[
   d(\psi \eta) = d\psi \wedge \eta + \psi \cdot d\eta
   \]

   is just \( d\eta \) on \( V \) and \( \text{supp}(\psi \eta) \subseteq U \).

   [Thus should be able to compute in local coor.]
On $\mathbb{R}^n$, these props completely det $\det dw$. For example,

$$d(dx_i \wedge \ldots \wedge dx_k) = 0 \quad \text{since} \quad d(dx_i) = 0$$

where here $x_i : \mathbb{R}^n \to \mathbb{R}$

and so if $\omega \in \Omega^2(\mathbb{R}^3)$ is

$$\omega = (x + z^2) \, dx \wedge dy + e^y \, dx \wedge dz$$

Then

$$d\omega = (d(x + z^2)) \wedge dx \wedge dy + (de^y) \wedge dx \wedge dz$$

$$= dx + 2z \, dz \wedge dx \wedge dy + e^y \, dy \wedge dx \wedge dz$$

$$= (2z - e^y) \, dx \wedge dy \wedge dz.$$

Two approaches to proving Thm:

A) Show well-defined for $\Omega^2(\mathbb{R}^n)$

Show that for $F : U \to V$ a diffeomorphism we have $F^*(d\omega) = d(F^*\omega)$. Define $d$ on $M$ via charts. This fact gives that it is well defined.

See text for details.
Coordinate-free definition: \( \omega \in \Omega^1(M) \)

Define \( dw \in \Omega^2(M) \) by the property that
if \( X, Y \in \mathfrak{X}(M) \) then
\[
dw(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])
\]
\[
f_{\lambda, \mu} \in C^0(M)
\]
Claim: \( f_{\lambda, \mu} \) only depends on \( X_\lambda \) and \( Y_\mu \).

Idea: Enough to show \( f_{\lambda, \mu} = \mu f_{\lambda, \nu} \),
which is an easy calculation.

In general, for \( \omega \in \Omega^k(M) \) define
\[
dw(X_0, X_1, \ldots, X_k) = \Sigma_{i=0}^{k} (-1)^i X_i \left( \omega(X_1, \ldots, \hat{X}_i, \ldots, X_k) \right)
\]
\[
+ \Sigma_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)
\]

For details on (A) and (B), see Lee.

Prop: \( F : M \to N \)
\( \omega \in \Omega^k(M) \).
Then
\[ d(F^\ast \omega) = F^\ast (dw) \]
Lie derivatives of differential forms: \( \forall \in \mathfrak{X}(M) \), \( \omega \in \Omega^k(M) \). Define \( L_V \omega \in \Omega^*(M) \) by

\[
(L_V \omega)_p = \frac{d}{dt} \bigg|_{t=0} (\Theta^*_t \omega)_p
\]

where \( \Theta \) is the flow assoc to \( V \). in \( \Lambda^k T_p M \)

Alternatively, using the same ideas as the proof that \( L_X Y = [X, Y] \) it follows that for any \( X_1, \ldots, X_k \in \mathfrak{X}(M) \)

\[
L_V \omega \left( X_1, \ldots, X_k \right) = V(\omega(X_1, \ldots, X_k)) - \omega([V, X_1], X_2, \ldots, X_k) - \ldots - \omega(X_1, X_2, \ldots, [V, X_k])
\]

Prop: \( \forall \in \mathfrak{X}(M) \) and \( \omega, \eta \in \Omega^*(M) \), then

\[
L_V (\omega \wedge \eta) = (L_V \omega) \wedge \eta + \omega \wedge (L_V \eta)
\]

Cartan's Magic Formula: \( L_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega) \)

Here, the interior product \( V \lrcorner \omega \) is the \( k-1 \) form defined by

\[
(V \lrcorner \omega)(w_1, \ldots, w_{k-1}) = \omega(V_p, w_1, \ldots, w_{k-1})
\]

\( \in T_p M \)