Integration on Manifolds (Lecture 30)

An orientation on $M^n$ is a locally consistent choice of orientation on each $T_pM$; it can be specified by $\omega \in \Omega^n(M)$ where each $\omega_p \neq 0$.

**Partition of Unity:** Countable $(U_i, \varphi_i, \psi_i)$ where $(U_i, \varphi_i)$ are smooth charts, $\psi_i : M \to [0, 1]$ smooth functions and $\cap \text{supp } \psi_i \subseteq U_i$; (6) Every $p$ is in finitely many $\text{supp } \psi_i$.

$\sum_i \psi_i = 1$.

A form $\omega \in \Omega^n(M)$ is compactly supported if $\text{supp } \omega = \{ p \in M \mid \omega_p \neq 0 \}$ is contained in a cpt set.

**Def:** $U \subseteq \mathbb{R}^n$. For compactly supported $\omega \in \Omega^n(U)$, write $\omega = \int f dx_1 \cdots dx_n$ where $f \in C^\infty(\mathbb{R}^n)$ is 0 outside $U$. Define

$$\int_U \omega = \int_U f dV = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n$$

where $\mathcal{T} [a_i, b_i]$ is any cpt box containing $\text{supp } f = \text{supp } \omega$. 
Suppose $M$ is an oriented $n$-manifold and $\omega \in \Omega^n(M)$ is compactly supported. Define

$$ \int_M \omega = \sum \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* (\psi_i \cdot \omega) $$

where $(U_i, \varphi_i, \psi_i)$ is a partition of unity where each $\varphi_i$ is orientation preserving. [Query: Why can we add the last condition?]

Note: Since supp $\psi_i \subseteq U_i$, the form $(\varphi_i^{-1})^* (\psi_i \omega)$ is smooth and compactly supported on $\varphi_i(U_i)$.

Thm. $\int_M \omega$ does not depend on the choice of the partition of unity.

Proof: First note that $\int_M \omega$ with respect to some fixed $(U_i, \varphi_i, \psi_i)$ is linear in $\omega$. Suppose $(V_j, \overline{\varphi}_j, \overline{\psi}_j)$ is another partition of 1. Consider

$$ \omega = \sum_{i,j} (\psi_i \overline{\psi}_j \omega) $$

[which is really a finite sum.]

Now we have $\text{supp } \eta_{ij} \subseteq U_i$ and $V_j$. By linearity we just need to check $\int_M \eta_{ij}$ is the same in both partitions of unity.
To see this, note

\((\psi_i^{-1})^* (\eta_{ij})\)

is

\((\overline{\psi}_j \circ \psi_i)^* \left( \left( \overline{\psi}_j^{-1} \right)^* \eta_{ij} \right)\)

and so the theorem follows from

**Lemma:** Suppose \(F : U \rightarrow V\) is an orient. pres. diffeo of open subsets of \(\mathbb{R}^n\). If \(\eta \in \Omega^n(V)\) is compactly supported then

\[ \int_V \eta = \int_U F^* \eta \]

**Idea:** Write \(\eta = f(x) \, dx_1 \wedge \ldots \wedge dx_n\), \(f \in C^\infty(V)\)

and \(F^* \eta = g(x) \, dx_1 \wedge \ldots \wedge dx_n\), \(g \in C^\infty(U)\)

Have

\[ g(x) = (F^* \eta) \left( \left. \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right|_x \right) \]

\[ = \eta_{\left. F(x) \right|} \left( dF \left( \left. \frac{\partial}{\partial x_1} \right|_x \right), \ldots, dF \left( \left. \frac{\partial}{\partial x_n} \right|_x \right) \right) \]

\[ = f(F(x)) \cdot \det(D_x F)^t = f(F(x)) \det(D_x F) \]

Since \(F\) is orient. pres, \(\det(D_x F) > 0\). So lemma

is now "just" the change of variables formula for integrals.
Volumes of Riemannian Manifolds:

$M^n$ oriented with Riemannian metric $g$.

The volume form $\omega_g \in \Omega^n(M)$ is defined by

$$\omega_g(v_1, \ldots, v_n) = \text{Signed vol of the parallelepiped spanned by } v_1, \ldots, v_n \in T_p M \text{ with respect to } g_p.$$  

$\Rightarrow \omega_g = 1$ on any pos.

orthonormal basis.

More explicitly

$$\omega_g(v_1, \ldots, v_n) = \sqrt{\det(g_p(v_i, v_j))}$$

if $(v_1, \ldots, v_n)$ are a pos. oriented basis.

Reason this is the right formula: On $\mathbb{R}^n$ with $g_{\cdot \cdot}$,

given $(v_1, \ldots, v_n)$ consider the matrix $A = (\begin{smallmatrix} v_1 & \cdots & v_n \end{smallmatrix})$

Then $\det(g_{\cdot \cdot}(v_i, v_j)) = \det(AA^t) = (\det A)^2$

$= (\text{sign vol span } v_i)^2$.

For $f \in C^\infty(M)$ can now define $\int_M f \, dV$

as $\int_M f \omega_g$. 