Lecture 29: Partitions of Unity

Goal: Define \( \int_M \) where \( \omega \in \Omega^n(M) \) and \( M \) is an \( n \)-manifold.

Def: The support of \( f: M \to \mathbb{R} \) is \( \text{supp } f = \{ p \in M \mid f(p) \neq 0 \} \).

HW #2: Given \( p \in M \) there is a smooth chart \((U, \varphi)\)
about \( p \) and a smooth \( f: M \to [0, 1] \) where

\( a \) \( f = 1 \) on a nbhd of \( p \)

\( b \) \( \text{supp } f \subseteq U \).

Thm: Every smooth \( M^n \) has a countable set of smooth charts \((U_i, \varphi_i)\) with bump functions \( f_i \) so that:

\( a \) \( \bigcup \text{supp } f_i = M \)

\( b \) Any \( p \in M \) is in finitely many \( \text{supp } f_i \).

Cor. Any smooth \( M^n \) has a Riemannian metric.
Pf of Cor: Let \((U_i, \varphi_i, f_i)\) be as in the
then. Define \((g_i)_p : T_p M \times T_p M \to \mathbb{R}\) by
\[
\begin{align*}
g_i(v_p, w_p) &= \begin{cases} 
0 & \text{if } p \notin \text{supp } f_i \\
\frac{f(p) g_{\varphi(p)}(d\varphi_i(v_p), d\varphi_i(w_p))}{\text{if } p \in \text{supp } f_i}
\end{cases}
\end{align*}
\]
[Query: Is \(g_i\) a Riemannian metric?]

Not quite a Riemannian metric,
satisfies everything except pos. def;
Do have \(g_i(v_p, v_p) \geq 0\) though.

Define \(g : T_p M \times T_p M\) by
\[
g(v_p, w_p) = \sum_i g_i(v_p, w_p)
\]
which makes sense because of \(\Box\). Its an actual
Riemannian metric since some term in \(g(v_p, v_p)\)
is \(\geq 0\).
Lemma: A smooth $M^n$ is a countable union of compact $K_i$ where $K_i \subseteq \text{Int}(K_{i+1})$.

[Query: What if $M = \mathbb{R}^n$?]

Pf: $M$ is second countable and locally compact. See A.60 in Lee.

Pf of Thm: Let $L_i = K_i \setminus \text{Int}(K_{i-1})$.

By compactness, there exist finitely many $(U_\alpha, p_\alpha, f_\alpha)$ so that

1. Each $U_\alpha \subseteq \text{Int}(K_{i+1}) \setminus K_{i-1}$
2. The support of $f_\alpha$ cover $L_i$

The union of these as $i$ varies is what we seek; condition 6 holds since the $U_\alpha$ for $L_i$ meet at most $L_{i+1}$ and $L_{i-1}$. 
Thm: Every smooth $M^n$ has a countable set of smooth charts $(U_i, \varphi_i)$ and $\psi_i \in C^\infty(M)$ where:

1. Any $p \in M$ is infinitely many $\text{supp } \psi_i$
2. $\text{supp } \psi_i \subseteq U_i$
3. For all $p \in M$ 
   \[ \sum_i \psi_i(p) = 1. \]

Pf: Let $(U_i, \varphi_i, f_i)$ be as in the first time.

Define $f \in C^\infty(M)$ by 
\[ f(p) = \sum_i f_i(p). \]

[Makes sense by ①]. Since $f(p) > 0$ [by ②] set 
\[ \psi_i = f_i / f. \] Then $\sum_i \psi_i = 1.$

Motivation: Break $\int_M \omega$ into $\int_M (\sum_i \psi_i) \omega = \sum_i \int_M \psi_i \omega$ and evaluate the latter as $\int_{\varphi(U_i)} (\varphi^{-1})^*(\psi_i \omega)$.