Lecture 37: Homotopies and Cohomology.

Smooth maps $F, G : M \to N$ are smoothly homotopic if $\exists H : M \times I \to N$ with $H \circ i_0 = F$ and $H \circ i_1 = G$.

Here $i_t : M \to M \times I$ is the inclusion at height $t$.

Thm. Homotopic maps induce the same map on cohomology.

Homotopy Operator Lemma Let $M^n$ be smooth w/o boundary. There exist linear maps $h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$ for all $k$ so that

$$d(h\alpha) + h(d\alpha) = \alpha_1 - \alpha_0 \quad \bigcirc$$

where $\alpha \in \Omega^k(M \times I)$ and $\alpha_t = i_t^*(\alpha)$.

In particular, if $\alpha$ is closed then $[\alpha_1] = [\alpha_0]$ in $H^k(M)$. This also implies the theorem by taking $\alpha = H^*w$ for $w \in \Omega^k(N)$ as then $\alpha_0 = F^*w$ and $\alpha_1 = G^*w$.

[Say why useful to have defined for non-closed forms.]
Pf: Define

\[(h\alpha)(V_1, \ldots, V_{k-1}) = \int_0^1 \alpha_{(m,t)}(\frac{\partial}{\partial t}, V_1, \ldots, V_{k-1}) \, dt \]

in \(T_{(m,t)} M \times I\)

To check \(\otimes\) at \(m \in M\), work in local coordinates \((x_1, \ldots, x_n)\).

By linearity of \(h\) and permuting coordinates, enough to check

\[\alpha = f(x,t) \, dx_1 \wedge \ldots \wedge dx_k\]

and

\[\beta = f(x,t) \, dt \wedge dx_1 \wedge \ldots \wedge dx_{k-1}\]

For \(\alpha\), note \(h\alpha = 0\) and \(h(d\alpha) =\)

\[h\left(\frac{\partial f}{\partial x}(x,t) \, dt \wedge dx_1 \wedge \ldots \wedge dx_k\right)\]

\[= \left(\int_0^1 \frac{\partial f}{\partial x}(x,t) \, dt\right) dx_1 \wedge \ldots \wedge dx_k\]

\[= (f(x,1) - f(x,0)) \, dx_1 \wedge \ldots \wedge dx_k = \alpha_1 - \alpha_0\]

For \(\beta\), work this out as problem #5 on HW 12.

\[\square\]

[So now we know \(H^* (\mathbb{R}^n)\), but still need better tools... But first, a concrete example.]

Thm: $H^k(S^1) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

Pf: Have $\tilde{V}: H'(S^1) \to \mathbb{R}$ given by $\tilde{V}([\omega]) = \int_{S^1} \omega$.

Suppose $[\omega] \in \ker \tilde{V}$. Use angle coordinates $\Theta$ and write $\omega = f \, d\Theta$ for $f \in C^\infty(S^1)$. Define $g \in C^\infty(S^1)$ by $g(\Theta) = \int_0^\Theta f(t) \, dt$; this makes sense because $\int_0^{2\pi} f \, dt = 0$. Now $dg = \omega$ and so $[\omega] = 0$ in $H'(S^1)$. So $H'(S^1) = \mathbb{R}$.

Called a retract of $D^2$ to $S^1$.

Cor: There is a smooth $R: \mathbb{R}^2 \to S^1$ with $R|_{S^1} = \text{id}_{S^1}$.

Pf: By HW12 #2, $R^*: H'(S^1) \to H'(\mathbb{R}^2)$ must be injective. But there's no such map $R \to 0$.

Def: Manifolds $M$ and $N$ are smoothly homotopy equivalent if $\exists$ smooth maps $F: M \to N$ and $G: N \to M$ with $G \circ F$ homotopic to $\text{id}_M$ and $F \circ G$ homotopic to $\text{id}_N$. 
\textbf{Ex.}: \( \mathbb{R}^n \) is homotopy equivalent to a point.

- \( M = \mathbb{R}^2 \setminus \{0\} \) is homotopy equivalent to \( N = S^1 \).

\[ F: M \to N \text{ is } F(x) = \frac{x}{1|x|} \]

\[ G: N \to M \text{ inclusion} \]

\underline{Check}: \( F \circ G = \text{id}_N \checkmark \)

\[ G \circ F \text{ is homotopic to } \text{id}_M \text{ via } H: M \times I \to M \]

given by \( t \cdot x + (1 - t) \cdot \frac{x}{1|x|} \).

\textbf{Cor.}: If \( M \) and \( N \) are homotopy equivalent, then \( H^*(M) \cong H^*(N) \).

\underline{Pf.}: Consider \( H^*(M) \xleftarrow{F^*} H^*(N) \). Now \( F^* \circ G^* \)

\[ = (G \circ F)^* = (\text{id}_M)^* = \text{id}_{H^*(M)} \text{. Similarly,} \]

\[ G^* \circ F^* = (F \circ G)^* = (\text{id}_N)^* = \text{id}_{H^*}(N) \text{.} \]

So \( F^* \) and \( G^* \) are inverse bijections. \( \square \)