Original goal: Do calculus on things locally like $\mathbb{R}^n$. 

[Derivatives, vector fields, differential forms, Lie bracket, Lie derivatives...]

So far our understanding of manifolds has been based on their local properties. Indeed, the coremost central concepts of this course have been how to define globally objects we understand on $\mathbb{R}^n$.

New focus: How do we use these tools to understand/distinguish the global topology of manifolds? 

\[
\begin{align*}
&\bigcirc \text{ vs. } \bigcirc \bigcirc \bigcirc \text{ vs. } \bigcirc \bigcirc \\
&\text{Showing: } \text{Existence can be easier than non-existence.}
\end{align*}
\]

Ex: $S^1 \times S^1 \subseteq \mathbb{R}^4$ is diffeomorphic to $\{(\sqrt{x^2+y^2}-2)^2+z^2=1\}$

\[
\{(w,v,w,x) \mid w^2+v^2=w^2+x^2=1\}
\]

Ex: $S^1$ and $S^3$ have nowhere-vanishing vector fields $S^1$ and $S^3$ has infinitesimal gen of mult by $e^{it}$. 

\[
\begin{align*}
C^2 &\rightarrow S^3 \quad \text{C^2} \rightarrow S^3 \\
\end{align*}
\]
Thm: $\emptyset$ and $\mathbb{S}^2$ are not diffeomorphic.

Thm: Every $x \in \mathcal{X}(\mathbb{S}^2)$ vanishes at at least one pt.

Thm: $D^n = \{ x \in \mathbb{R}^n \mid 1 \times 1 \leq 1 \}$. There does not exist a smooth map $\phi : D^n \to \mathcal{D}^n$ where $f|\partial D^n = \text{id}|\partial D^n$.

[Of course the distinction between existent and non-existence is fuzzy.]

Thm: Suppose $F : D^n \to D^n$ is smooth. Then $\exists \, p \in D$ with $F(p) = p$.

---

A form $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$; it is exact if $\exists \, \eta \in \Omega^{k-1}$ with $d\eta = \omega$.

Note: Since $d \circ d = 0$ have:

1. On $\mathbb{R}^2$: let
   
   $\omega = 2xy \, dx + x^2 \, dy$

   is closed and exact since $\omega = d(x^2y)$.

   The form $w = y \, dx$ is neither closed or exact.
2. Suppose $M \subseteq \mathbb{R}^3$. 
\[ \Omega^1(M) \leftrightarrow \mathfrak{X}(M) \]
\[ w = a \, dx + b \, dy + c \, dz \leftrightarrow a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} = V \]
\[ w \text{ closed } \iff \text{curl} \, V = 0 \]
\[ w \text{ exact } \iff \text{grad} \, f \text{ for } f \in C^0(M) \]

Forshadowing: When $M = \mathbb{R}^3$ then closed and exact are equivalent.

3. Suppose $M$ has no boundary. Then $\omega \in \Omega^n(M)$ is closed and orientable.

and if $\omega$ is exact then $\int_M \omega = \int_{\partial M} \eta = 0$

If $\omega$ is the volume form on $S^2$, then $\int_{S^2} \omega = 4\pi$ and so $\omega$ is not exact.

Suppose $\omega \in \Omega^k(M)$ is closed but not exact. Then so is $\omega + \text{d}\eta$ for every $\eta \in \Omega^{k-1}(M)$.

Note: \{exact\} \subseteq \{closed\} \subseteq \Omega^k(M) are linear subspaces, so can define the $k^{th}$-deRham cohomology group as
\[ H^k(M) = \frac{\{\text{closed}\}}{\{\text{exact}\}} \]

That is, $H^k(M)$ is the set of equiv. classes $[\omega]$ when $\text{d} \omega = 0$ and $[\omega] = [\omega']$ if $\omega - \omega'$ is exact.

Fact: When $M$ iscpt, this actually a finite-dimensional
vector space.

Note if you know about simplicial/singular/cellular cohomology, this is $H^k(M; \mathbb{R})$.  

\[ H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k > 0 \\ \mathbb{R} & k = 0 \end{cases} \]

Ex: Suppose $M$ is a compact connected orientable $n$-mfd.

Then $H^n(M) \cong H^0(M) \cong \mathbb{R}$.

Ex: 

\[ \begin{array}{c|c|c|c|c|c} 
\text{Ex} & \mathbb{R}^2 & \mathbb{R}^4 & \mathbb{R}^6 \\
H^1 & 0 & \mathbb{R}^2 & \mathbb{R}^4 & \mathbb{R}^6 \\
\end{array} \]

---

Basic properties:

1. Set $H^*(M) = \bigoplus_{k=0}^n H^k(M)$. Then $H^*(M)$ is an algebra with the multiplication $[\alpha] \cdot [\beta] = [\alpha \wedge \beta]$.

   Makes sense because $(\alpha + d\eta) \wedge \beta = \alpha \wedge \beta + d\eta \wedge \beta = \alpha \wedge \beta + d(\eta \wedge \beta)$ since $d\beta = 0$; same for other poss. reps of $[\beta]$.

2. If $F: M \to N$ get a map $F^*: H^*(N) \to H^*(M)$ from the associated map $\Sigma^*(N) \to \Sigma^*(M)$ since $d$ and $F$ commute.
3. Cor: If $M$ and $N$ are diffeomorphic, then

$$H^*(M) \cong H^*(N) \quad (\text{as R-algebras}).$$

4. Prop: Suppose $M_1, \ldots, M_n$ are smooth manifolds. Then

$$H^*(\bigotimes_{i=1}^n M_i) \cong \left( \prod_{i=1}^n H^*(M_i) \right).$$

5. Prop: If $M^n$ is connected then $H^0(M) = \mathbb{R}$.

Pf: There are no exact forms since $\Omega^{-1} = 0$. Then $A$

function $f \in \Omega^0(M)$ has $df = 0$ exactly

when it is constant. The subspace of

constant functions is $\cong \mathbb{R}$. \qed