Thm: \( M \) smooth. There are unique maps \( d: \Omega^k(M) \to \Omega^{k+1}(M) \) satisfying:

1. \( d \) is linear/IR
2. \( \omega \in \Omega^k(M) \) and \( \eta \in \Omega^l(M) \):
   \[
d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta
   \]
3. \( \circ \) \( d \circ d = 0 \)
4. For \( f \in \Omega^0(M) = C^\infty(M) \), the \( df \in \Omega^1(M) \) is the usual differential, i.e. \( df(v_p) = v_p f \).

One reason for \((-1)^k\) in 2: Needed so that

\[
d(\eta \wedge \omega) = (-1)^k \eta \wedge d\omega.
\]

Lie derivatives: \( \nu \in \mathfrak{X}(M) \) and \( \omega \in \Omega^k(M) \).

Define \( \mathcal{L}_\nu \omega \in \Omega^k(M) \) by

\[
(\mathcal{L}_\nu \omega)_p := \left. \frac{d}{dt} \left( \Theta^*_t \omega \right) \right|_{t=0}
\]

in \( \bigwedge^k T_p M \)

where \( \Theta \) is the flow associated to \( \nu \).

Using the proof that \( \mathcal{L}_X Y = [X,Y] \)
this is the same as defining $\mathcal{L}_V \omega$ by the
property that for any $X_1, \ldots, X_k \in \mathcal{X}(M)$ one has

\[
\mathcal{L}_V \omega(X_1, \ldots, X_k) = V(\omega(X_1, \ldots, X_k)) \\
- \omega([V, X_1], X_2, \ldots, X_k) - \cdots \\
- \omega(X_1, X_2, \ldots, [V, X_k])
\]

Prop: $V \in \mathcal{X}(M)$ and $\omega, \eta \in \Omega^*(M)$. Then

\[
\mathcal{L}_V (\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta)
\]

[No signs needed here as $\mathcal{L}_V$ doesn't change the]
[degree of the forms, $\mathcal{L}_V$ and $d$ are related!]

Cartan's Magic Formula:

\[
\mathcal{L}_V \omega = V \lrcorner dw + d(V \lrcorner \omega)
\]

Here, $V \lrcorner \eta$ is the \underline{interior product}
defined by

\[
(V \lrcorner \eta)(W_1, \ldots, W_{k-1}) = \eta(V, W_1, \ldots, W_{k-1})
\]

$\in T_p M$
Stokes Thm: Let $M$ be an oriented smooth $n$-manifold with boundary. If $\omega \in \Omega^n(M)$ is compactly supported, then $\int_M d\omega = \int_{\partial M} \omega$.

Recall: Such an $M$ has charts to open sets in $\mathbb{R}^n$ and $H^n = \{ x \in \mathbb{R}^n \mid x_n > 0 \}$, $\{ x \in H^n \mid x_n = 0 \}$, $\mathbb{R}^n$. Then $\partial M = \{ p \in M \mid \exists$ smooth $(U, \varphi)$ with $\varphi(p) \in \partial H^n \}$.

Note $\partial M$ is itself a manifold (with the topology inherited from $M$) without boundary.

Prop: An orientation of $M$ induces one of $\partial M$. [In particular, $\partial M$ is orient when $M$ is.]

For $p \in \partial M$, the tangent space $T_pM$ is still $\mathbb{R}^n$; you can view it as isometric with $T_pH^n = T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$. [Alt, its derivations $\partial_i$ smooth fns at $p$.]
Given \( v \in T_p M \) for \( p \in \partial M \) have one of \( 0 \) \( v \in T_p \partial M \) \( 0 \) \( v \) inward pointing \( 0 \) outward pointing.

Given an orientation of \( M \), orient \( \partial M \) by the following rule. A basis \( b_1, \ldots, b_{n-1} \in T_p \partial M \) is positively oriented if when \( n_p \in T_p M \) is outward pointing, then \( n_p, b_1, \ldots, b_{n-1} \) is a pos. basis for \( T_p M \).

**Ex:**

![Diagram](image1)

**Ex:**

![Diagram](image2)

So \( \partial H^n \) gets the standard orient of \( \mathbb{R}^{n-1} \) when \( n \) is even and the opposite orient. when \( n \) is odd.

To prove the prop, have to check that the pointwise orient defined above is locally consistent. But that's clear from the picture for \( H^n \).
Special cases:

1. If \( \partial M = \emptyset \), view \( \int_{\partial M} w \) as 0.

2. If \( \dim M = 1 \), then \( \partial M = \) some points.

   An orient of a point mfld \( p \) is just a sign \( \varepsilon_p = \pm 1 \) and \( \int_{p} f = \varepsilon_p f(p) \).

3. So if \( M = [a, b] \) and \( f \in C^0(M) \) then

   \[ \int_{M} df = \int_{a}^{b} f'(t) \, dt = f(b) - f(a) = \int_{\partial M} f \]

[If time remains, blather about how this connects to Math 241]