Lecture 40: Degrees of maps of spheres.

Last time: \( H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases} \)

The degree of a smooth \( F: S^n \to S^n \) is the \( \deg(f) \in \mathbb{R} \) so that \( F^*([w]) = \deg(f)[w] \) for all \( [w] \in H^n(S^n) \cong \mathbb{R} \).

Ex: \( \deg(id_{S^n}) = 1 \) \( \deg(\text{const map}) = 0 \)

\( \circ \) \( \deg(\text{Reflect in } \mathbb{R}^n) = -1 \)

\( S^n \subseteq \mathbb{R}^{n+1} \)

\( F(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1}) \)

\( F^*(\text{Volume form } w_g) = -w_g \)

\( \circ \) \( S^1 \to S^1 \quad d\Theta \) gen of \( H^1(S^1) \)

\( F^*(d\Theta) = 2d\Theta \implies \deg F = 2 \)

\( \circ \) \( S^1 \to S^1 \) has \( F^*(d\Theta) = n d\Theta \implies \deg F = n \)
Note: Lack of understanding in that $F^*(w) = (\deg f)w$; typically this only happens on the level of cohomology.

$F: S^1 \rightarrow S^1$

$F^*(d\theta) = g(\theta) d\theta$

Note $H^n(S^n) \xrightarrow{\cong} \mathbb{R}$

$[w] \mapsto \int_{S^n} w$

So if $[w] \neq 0$ have:

$$\deg F = \frac{\int_{S^n} F^* w}{\int_{S^n} w}$$

In this case $F$ is homotopic to $id \Rightarrow \deg = 1$.

Thm: For any smooth $F: S^n \rightarrow S^n$, $\deg F \in \mathbb{Z}$.

For any regular value $q \in S^n$ we have

$$\deg F = \sum_{p \in F^{-1}(q)} \begin{cases} +1 & \text{if } df_p \text{ is orient. pres.} \\ -1 & \text{if } df_p \text{ is orient. rev.} \end{cases}$$
Note: Homotopic maps have the same degree.

In fact, the converse is true as well.

**Proof:** Let \( q \in S^n \) be a regular value of \( F \). Then \( F^{-1}(q) \) is an embedded submanifold of \( d \mu = 0 \), i.e. a finite set of points \( p_1, \ldots, p_k \). Can choose disjoint open nbhds \( U_i \) of \( p_i \) and \( V \) of \( q \) so that each \( F|_{U_i} \) is a diffeo onto \( V \).

Pick \( \omega \in \Omega^2(S^n) \) with \( \text{supp} \, \omega \subseteq V \) and \( \int_V \omega = 1 \).

Then

\[
\deg F = \int_{S^n} F^*\omega = \sum_{i=1}^k \int_{U_i} (F|_{U_i})^*\omega
\]

\[
= \sum_{i=1}^k (\int_V \omega) (1 \text{ if } F|_{U_i} \text{ preserves orient} -1 \text{ if } F|_{U_i} \text{ reverses orient})
\]

\[
= \sum_{i=1}^k \begin{cases} +1 & \text{if } dF_{p_i} \text{ preserves orient} \\ -1 & \text{if } dF_{p_i} \text{ reverses orient} \end{cases}
\]

\[\square\]
Properties: (a) For $F, G : S^n \to S^n$ have
\[
\text{deg} (F \circ G) = (\text{deg} F)(\text{deg} G)
\]

Proof: 
\[
\text{deg} (F \circ G)[w] = (F \circ G)^*[w] = G^*(F^*[w])
\]
\[
= G^* (\text{deg} F)[w] = (\text{deg} G) (\text{deg} F)[w]
\]

(b) If $A : S^n \to S^n$ is the antipodal map $x \mapsto -x$
then $\text{deg} A = (-1)^{n+1}$ since $A$ is the composition
of $n+1$ reflections.

(c) If $F : S^n \to S^n$ has no fixed points, then $\text{deg} F = (-1)^{n+1}$

Proof: Since $F(x) \neq x$, the line segment joining
$-x$ to $F(x)$ does not go through $0$.
So $H : S^n \times I \to S^n$ given by
\[
H(x,t) = \frac{(1-t)F(x) - tx}{\| (1-t)F(x) - tx \|}
\]
makes sense and shows that $F$ is homotopic to $A$.
So $\text{deg} F = \text{deg} A = (-1)^{n+1}$
Thm \( S^n \) has a nowhere vanishing vector field
iff \( n \) is odd

\[ \begin{align*}
\text{Pf:} & \quad \text{Suppose } X \in \mathcal{X}(S^n) \text{ is nowhere vanishing. Let } \\
& \quad \Theta_t \text{ be the associated flow. Choose } \epsilon > 0 \text{ so that } \\
& \quad \Theta_\epsilon \text{ has no fixed points (can do since } X_p \neq 0 \text{ and } S^n \text{ is cpt.}) \\
& \quad \text{Thus } \deg(\Theta_\epsilon) = (-1)^{n+1}. \text{ As } \Theta_\epsilon \text{ is homotopic to } \text{id}_{S^n} \\
& \quad (\text{via } \Theta_t) \text{ have } \deg \Theta_\epsilon = \deg \text{id}_{S^n} = 1. \text{ So } n \text{ is odd.}
\end{align*} \]

When \( n \) is odd, you constructed nowhere vanishing vector fields on the HW. Eg. \( S^n \subseteq \mathbb{R}^{n+1} \subseteq \mathbb{C}^{\frac{n+1}{2}} \)
and consider the flow \( \Theta_t(z) = e^{it}z \).