Poincaré Duality: Suppose $M$ is a compact oriented $n$-manifold without boundary. Then $H^k(M) \cong H^{n-k}(M)$ for $0 \leq k \leq n$.

**Ex.** $H^n(M) \cong H^0(M) \cong \mathbb{R}$.

**Ex:** $H^*(S^1 \times S^2) = \mathbb{R}^4$.

**Thm:** $M$ as above. Then $H^k(M) \times H^{n-k}(M) \to \mathbb{R}$ is a non-degenerate bilinear form. [Query: What does non-degenerate mean?]

**Notes:** @ Well-defined since by HW have that $[\alpha] \wedge [\beta] \in H^n(M)$ can be defined as $[\alpha \wedge \beta]$ and $\int_M \omega$ depends only on $[\omega] \in H^n(M)$ by Stokes' thm.
6. By $H^i, H^k, (M)$ and $H^{n-k}, (M)$ are finite dim'l.
   As $B$ is non-degenerate, have $H^k(M) \rightarrow (H^{n-k}, (M))^*$
   and $H^{n-k}(M) \rightarrow (H^k(M))^*$. Thus $H^k(M) \cong H^{n-k}, (M)$.

© This version is interesting even for surfaces.

$T = \bigcirc = S^1 \times S^1$ \hspace{1cm} $H^1(T) = \mathbb{R}^2$ with basis $[d\theta_1]$ and $[d\theta_2]$

$B([d\theta_1], [d\theta_1]) = 0$
$B([d\theta_1], [d\theta_2]) = 4\pi^2.$

© $T^n = S^1 \times \ldots \times S^1$. Then $H^k(T^n)$ has

basis $[d\theta_{i_1}, \ldots, d\theta_{i_k}]$ where $1 \leq i_1 < \ldots < i_k \leq n$.

In particular \hspace{1cm} $\dim H^k(T^n) = \binom{n}{k} = \binom{n}{n-k} = \dim H^{n-k}(T^n)$.

Where does Poincaré Duality come from? Initially
this seems like an indivisible statement...
A closed manifold can't be divided into smaller submanifolds
(although do have connected sums...) and it fails
for $\mathbb{R}^n$: $H^n(\mathbb{R}^n) \neq H^0(\mathbb{R}^n)$.

Thm: $M$ an oriented $m$-mfd. Then

$$H^k(M) \times H_c^{n-k}(M) \rightarrow \mathbb{R}$$

$$\left[ \alpha \right] \left[ \beta \right] \mapsto \int_M \alpha \wedge \beta$$

is non-degenerate.

Note: When $M$ is compact, $H_c^{n-k}(M) = H^{n-k}(M)$
so this is a generalization of the earlier theorem.

Can check directly that $H_c^i(\mathbb{R}^n) = \begin{cases} \mathbb{R} & i = n \\ 0 & \text{otherwise} \end{cases}$

You did $i = 0$ on HW. Here's a sketch for $H_c^1(\mathbb{R})$.

Take $w \in \Omega^1_c(\mathbb{R})$, say $w = f(t) \, dt$.

$$f(t)$$

$\int_0^1 f(t) \, dt$

$\exists g \in C^\infty(\mathbb{R})$ with $dg = w$

but any such $g$ is

not compactly supported; more precisely

$$g(t) = \int_{-\infty}^t f(s) \, ds$$
Alternatively, again have \( H_c^n(\mathbb{R}^n) \to \mathbb{R} \) given by \([w] \mapsto \int_{\mathbb{R}^n} w\).

One proof of Poincaré Duality is inductive, starting from the case of \( \mathbb{R}^n \) and using M-V (which also exists for \( H_c^* \)). See Bott & Tu for details.

Suppose \( S \subseteq M \) is a closed oriented submanifold of \( M \). Then get a homomorphism \( H^k(M) \to \mathbb{R} \). Since \((H^k(M))^* \cong H^{n-k}(M)\) under B, there must exist \([\beta] \in H^{n-k}(M)\) so that \( \int_M [\alpha] \wedge [\beta] = \int_S [\alpha] \) for all \([\alpha] \in H^k(M)\). The class \([\beta]\) is called the Poincaré dual of \( S \).

[Secretly, Poincaré Duality is a relationship between cohomology and homology...\]
\[ \text{Ex: } T = S^1 \times S^1 \quad S = S^1 \times \{1\} \]

1st guess: \( \beta = [d\theta_1] \)

\[ \int_M [d\theta_1] \wedge [d\theta_1] = 0 \neq \int_S [d\theta_1] = 2\pi \]

2nd guess: \( [\beta] = \frac{1}{2\pi} [d\theta_2] \). This works since

and \( [\omega] = a [d\theta_1] + b [d\theta_2] \) and so

\[ \int_M [\omega] \wedge [\beta] = \int_M a [d\theta_1] \wedge [\beta] = 2\pi a \]

and \( \int_S \omega = \int_S a \, d\theta_1 = 2\pi a \checkmark \)

**Thm.** Suppose \( S \) and \( S' \) are closed orient submanifolds of closed orient \( M^n \) of dim \( k \) and \( n-k \).

Then if \( S \) and \( S' \) are transverse,

\[ \#(S \cap S') = \int_M [\beta_S] \wedge [\beta_{S'}] \]

with signs \( \checkmark \)