Lecture 3: Smooth maps and diffeomorphisms.

Last time:

**Smooth manifold:** A topological manifold $M$ with charts covering $M$ so that each pair is compatible, i.e.

\[ \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \]

is a diffeomorphism.

**Smooth fn:** $M$ a smooth mfd. A fn $f : M \to \mathbb{R}^k$ is smooth if for every smooth chart $(U, \varphi)$ the fn $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth.

\[ f = f \circ \varphi^{-1} \text{ coordinate rep'n.} \]

Reminders:

- No class on Monday
- HW #1 due Wednesday, Sept 3.
Lemma: $M$ smooth, $f: M \to \mathbb{R}^k$. If every $p \in M$ is contained in a smooth chart where $\hat{f}$ is smooth, then $f$ is smooth.

Pf: Given an arbitrary smooth chart $(U, \varphi)$ need to show $\hat{f}$ is smooth. Since smoothness is local, focus on $x \in \varphi(U)$. Let $(V, \psi)$ be a smooth chart where $\varphi^{-1}(x) \in V$ and $f \circ \varphi^{-1}$ is smooth. On $\varphi(U \cap V)$, we have

$$f \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})$$

smooth  smooth

So $\hat{f}$ is smooth.

Def: A continuous $f: M \to N$ between smooth manifolds is smooth if for all smooth charts $(U, \varphi)$ of $M$ and $(V, \psi)$ of $N$ the fn:

$$\psi \circ f \circ \varphi^{-1}: \varphi(f^{-1}(V) \cap U) \to \psi(V)$$

is smooth,
Note: If $X \subseteq \mathbb{R}^n$ is any set, we say $f: \mathbb{R}^n \to \mathbb{R}^m$ is smooth if $\exists$ open $U \supseteq X$ and a smooth function $\overline{f}: U \to \mathbb{R}^m$ where $\overline{f}|_X = f$.

Need this for manifolds with boundary.
Lemma [Lee pgs 34-36] Equivalently, a function \( f: M \to N \) is smooth if \( \forall p \in M \) there are charts \( (U, \varphi) \) of \( M \) and \( (V, \psi) \) of \( N \) so that
\begin{enumerate}
  \item \( p \in U \) and \( f(U) \subseteq V \)
  \item \( \psi \circ f \circ \varphi^{-1}: \varphi(U) \to \psi(V) \)
\end{enumerate}
is smooth.

[In this version, \( f \) is not assumed to be continuous]

Def: A \underline{diffeomorphism} between smooth manifolds \( M \) and \( N \) is a bijection \( f: M \to N \) where \( f \) and \( f^{-1} \) are both smooth.

Fact: There are 28 smooth 7-mfolds \( M_1, \ldots, M_{27} \) so no pair are diffeomorphic but each is homeomorphic to \( S^7 \).

Ex: \( N_1 = \mathbb{R} \) with \( A_1 = \left\{ (U, \varphi) \mid U = \mathbb{R}^2, \varphi = \text{id} \right\} \)

\[ N_2 = \mathbb{R} \] with \( A_2 = \left\{ (V, \psi) \mid V = \mathbb{R}^2, \psi = \frac{x^2}{3} \right\} \)
$A_1 \neq A_2$ since $f : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto x^{1/3}$

is not smooth w.r.t. $A_1$, but is smooth w.r.t. $A_2$.

$f : x \mapsto x^{1/3}$

But, $N_1$ and $N_2$ are diffeomorphic, via

$h : N_1 \rightarrow N_2$. [Check this!]

$x \mapsto x^3$

$\varphi\big|_{x \mapsto x}$

$\psi\big|_{x \mapsto x^{1/3}}$

$f \circ \varphi^{-1} : x \mapsto x^{1/3}$

$f \circ \psi^{-1} = x \mapsto (x^3)^{1/3} = x$.

Remember goal: Do calculus. Somehow, I've managed to define smooth fns w/o saying what their derivatives are.

What is the derivative of $f$ at $t = 0$?
Think back: $f: \mathbb{R}^n \to \mathbb{R}^k$ smooth. For $p \in \mathbb{R}^n$

$D_p f$ is the linear trans $\mathbb{R}^n \to \mathbb{R}^k$ best approx of $f$ near $p$, that is

$f(p + v) = f(p) + (D_p f).v + O(\|v\|^2)$

Concretely

$D_p f = \left( \begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_n} \end{array} \right)$

where $f = (f_1, \ldots, f_k)$. Basic idea for mfdls:

First need to figure out what $T_p M$ is.
Ex: $T_a \mathbb{R}^n = \{a + \lambda v \mid \lambda \in \mathbb{R}, v \in \mathbb{R}^n\}$

(a, v) + (a, v') = (a, v + v')

(a, v) + (b, w) = Nothing in particular

Now, view $D_p f: T_p \mathbb{R}^n \to T_{f(p)} \mathbb{R}^k$

Ex: $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$

$T_p S^2 = \{ (p, v) \in T_p \mathbb{R}^3 \mid v \cdot p = 0 \}$

What to do in general? At least 3 ways to do this...

Suppose $f : (U \subseteq \mathbb{R}^n) \to \mathbb{R}, \ a \in U$

Recall

directional derivative of $f$ at $a$ in direction $u$

\[
\frac{d}{dt} (f(a + tv)) \bigg|_{t=0} = (D_a f) \cdot v
\]

\[\nabla f(a)\]

Let $C^\infty(\mathbb{R}^n)$ denote the set of smooth functions $\mathbb{R}^n \to \mathbb{R}$. Consider

$D_{(a, v)} \mathcal{D}: C^\infty(\mathbb{R}^n) \to \mathbb{R}$
Note: (a) \( D_{(a,v)}(cf + g) = c\, D_{(a,v)}f + D_{(a,v)}g \)
for all \( f, g \in C^\infty(\mathbb{R}^n) \), \( c \in \mathbb{R} \)
(b) \( D_{(a,v)}(fg) = f(a)\, D_{(a,v)}g + g(a)\, D_{(a,v)}f \)

A map \( w : C^\infty(\mathbb{R}) \to \mathbb{R} \) is called a derivation at \( a \) if (a) \( w \) is \( \mathbb{R} \)-linear
(b) \( w(fg) = f(a)\, w(g) + g(a)\, w(f) \)

\( \mathcal{D}_a = \) set of all derivations at \( a \)

Prop: \( T^*a : \mathbb{R}^n \to \mathcal{D}_a \)

\[(a,v) \mapsto D_{(a,v)}\]

is an isomorphism of vector spaces.

Tricky bit: That this is onto.