Math 518: HW 8 due Wednesday, October 29, 2014.

1. Problem 11–4 of Lee on pages 299–300. You may assume that $M$ has no boundary.

2. Let $G = \text{GL}_n\mathbb{R}$. Define $n^2$ covector fields $\sigma_{ij}$ on $G$ for $1 \leq i, j \leq n$ as follows. At a point $X = (x_{ij})$ in $G$, define

$$\sigma_{ij} = \sum_{k=1}^{n} y_{ik} \, dx_{kj} \quad \text{where } Y = (y_{ik}) \text{ is the inverse matrix to } X.$$ 

Since taking the inverse is a smooth function on the Lie group $G$, the $\sigma_{ij}$ are in $\Omega^1(M)$.

(a) Prove that each $\sigma_{ij}$ is left-invariant. That is, for every $A \in G$, prove that $(L_A)^*(\sigma_{ij}) = \sigma_{ij}$.

(b) Prove that at every $A \in G$, the 1-forms $\sigma_{ij}$ form a basis for $T_A^*G$.

(c) Now take $n = 2$ and consider the circle $C \subset G$ which is the image of $\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for $t \in [0, 2\pi]$. Calculate each of the integrals $\int_C \sigma_{ij} = \int_0^{2\pi} \gamma^*(\sigma_{ij})$.


4. Let $g$ be the round Riemannian metric on $S^2$, that is, $g_p: T_pS^2 \times T_pS^2 \to \mathbb{R}$ is the restriction to $T_pS^2 \subset T_p\mathbb{R}^3$ of the usual dot product on $\mathbb{R}^3$, where as always $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1 \}$. Compute the coordinate form of $g$ with respect to each of the following charts. That is, for a chart $(U, \phi)$ compute the four functions $g_{ij}: \phi(U) \to \mathbb{R}$ so that

$$(\phi_*g)(x_1, x_2) = \sum_{i,j=1}^{2} g_{ij}(x_1, x_2) \cdot (dx_i \otimes dx_j)$$

(a) Sterographic projection away from $(0, 0, 1)$.

(b) Projection of the open upper hemisphere of $S^2$ onto the $xy$-plane.

In which of these coordinates do the angles implicitly assigned by $\phi_*g$ agree with the ordinary Euclidean angles on $\mathbb{R}^2$?

5. Problem 13–21 of Lee on page 347.

6. Problem 12–7 of Lee on page 325.