1. Consider the region $D$ in $\mathbb{R}^3$ bounded by the $xy$-plane and the surface $x^2 + y^2 + z = 1$.

(a) Make a sketch of $D$.

**Solution.** The sketch of $D$ is shown below.

(b) The boundary of $D$, denoted $\partial D$, has two parts: the curved top $S_1$ and the flat bottom $S_2$. Parameterize $S_1$ and calculate the flux of $F = (0, 0, z)$ through $S_1$ with respect to the upward pointing unit normal vector field. Check your answer with the instructor.

**Solution.** To parametrize $S_1$, one has

$$
\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 - u^2 \rangle, \quad 0 \leq u \leq 1, \ 0 \leq v \leq 2\pi.
$$

In order to calculate the flux, first we have

$$
\mathbf{r}_u = \langle \cos v, \sin v, -2u \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 0 \rangle,
$$

and so

$$
\mathbf{r}_u \times \mathbf{r}_v = \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle.
$$

Therefore, the flux of $F = (0, 0, z)$ through $S_1$ with respect to the upward pointing unit normal vector field is

$$
\int_{S_1} F \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^1 \mathbf{F}(u, v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dudv
$$

$$
= \int_0^{2\pi} \int_0^1 (0, 0, 1 - u^2) \cdot \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle \, dudv
$$

$$
= \int_0^{2\pi} \int_0^1 (1 - u^2)u \, dudv = \frac{\pi}{2}.
$$
(c) Without doing the full calculation, determine the flux of $F$ through $S_2$ with the downward pointing normals.

**Solution.** Since $F = 0$ on $S_2$, we know the flux of $F$ through $S_2$ is

$$\iint_{S_2} F \cdot n \, dS = \iint_{S_2} 0 \cdot n \, dS = 0.$$  

(d) Determine the flux of $F$ through $\partial D$ with the outward pointing normals.

**Solution.** By adding up the result from (a) and (b), one gets

$$\iint_{\partial D} F \cdot n \, dS = \iint_{S_1} F \cdot n \, dS + \iint_{S_2} F \cdot n \, dS = \frac{\pi}{2}.$$

(e) Apply the Divergence Theorem and your answer in (d) to find the volume of $D$. Check your answer with the instructor.

**Solution.** By the Divergence Theorem, one has

$$\iiint_D \text{div} F \, dV = \iint_{\partial D} F \cdot n \, dS.$$

Since $\text{div} F = 1$, one gets

$$\text{Volume}(D) = \iiint_D 1 \, dV = \iint_{\partial D} F \cdot n \, dS = \frac{\pi}{2}.$$

2. Consider the vector field $F = (-y, x, z)$.

(a) Compute curl $F$.

**Solution.**

$$\text{curl } F = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{pmatrix} = (0, 0, 2).$$

(b) For the surface $S_1$ above, evaluate $\iint_{S_1} (\text{curl } F) \cdot n \, dA$.

**Solution.** By applying the parametrization of $S_1$ in 1(b), one gets

$$\iint_{S_1} (\text{curl } F) \cdot n \, dA = \int_0^{2\pi} \int_0^1 (\text{curl } F) \cdot (r_u \times r_v) \, dudv$$

$$= \int_0^{2\pi} \int_0^1 2u \, dudv = 2\pi.$$
3. If time remains:

(a) Check your answer in 1(e) by directly calculating the volume of \(D\).

**Solution.** One can use the polar coordinate to calculate the volume of \(D\) in 1(e). Let

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.
\]

Then

\[
\text{Volume}(D) = \iint_{x^2+y^2 \leq 1} (1 - x^2 - y^2) \, dA = \int_0^{2\pi} \int_0^1 (1 - r^2) \, r \, dr \, d\theta = \frac{\pi}{2}.
\]

(b) Repeat 2 (b) for the surface \(S_2\) and also for the surface \(\partial D\).

**Solution.** The normal vector of \(S_2\) pointing downward is \(\mathbf{n} = -\mathbf{k}\). Thus,

\[
\iint_{S_2} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA = \iint_{S_2} (0,0,2) \cdot (0,0,-1) \, dA = -2 \iint_{S_2} dA = -2\pi.
\]

For the surface \(\partial D\) we get our answer by adding up the two integrals

\[
\iint_{\partial D} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA = \iint_{S_1} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA + \iint_{S_2} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA = 2\pi + (-2\pi) = 0.
\]

(c) For the vector field \(\mathbf{F} = (-y, x, z)\) from the second problem, compute \(\text{div}(\text{curl} \, \mathbf{F})\). Now suppose \(\mathbf{F} = (F_1, F_2, F_3)\) is an arbitrary vector field. Can you say anything about the function \(\text{div}(\text{curl} \, \mathbf{F})\)?

**Solution.** We already know, in 2(a), that \(\text{curl} \, \mathbf{F} = (0,0,2)\), so \(\text{div}(\text{curl} \, \mathbf{F}) = 0\). Generally, suppose suppose \(\mathbf{F} = (F_1, F_2, F_3)\) is an arbitrary vector field. Then

\[
\text{curl} \, \mathbf{F} = \det \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial }{\partial x} & \frac{\partial }{\partial y} & \frac{\partial }{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} , \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} , \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right),
\]

and

\[
\text{div}(\text{curl} \, \mathbf{F}) = \frac{\partial }{\partial x} \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial }{\partial y} \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial }{\partial z} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]

\[
= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.
\]