1. Consider the region $R$ in $\mathbb{R}^2$ shown below at right. In this problem, you will do a change of coordinates to evaluate:

$$\iint_R x - 2y \, dA$$

(a) Find a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which takes the unit square $S$ to $R$.

Write you answer both as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and as $T(u, v) = (au + bv, cu + dv)$, and check your answer with the instructor.

**SOLUTION:**

$T(u, v) = (2u + v, u + 3v)$. In matrix form,

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

(b) Compute $\iint_R x - 2y \, dA$ by relating it to an integral over $S$ and evaluating that. Check your answer with the instructor.

**SOLUTION:**

The Jacobian of $T$ is

$$\det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 6 - 1 = 5$$

So

$$\iint_R x - 2y \, dA = \iint_S [(2u + v) - 2(u + 3v)]5 \, dA$$

$$= \int_0^1 \int_0^1 -25v \, du \, dv = \left[ \frac{-25v^2}{2} \right]_0^1 = -\frac{25}{2}$$
Another simple type of transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, which has the general form $T(u, v) = (u + a, v + b)$ for a fixed $a$ and $b$.

(a) If $T$ is a translation, what is its Jacobian matrix? How does it distort area?

**SOLUTION:**

If $T(u, v) = (u + a, v + b)$ where $a$ and $b$ are constants, then the Jacobian is

$$ det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1. $$

So $T$ does not distort areas.

(b) Consider the region $S = \{u^2 + v^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates $(u, v)$, and the region $R = \{(x-2)^2 + (y-1)^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates $(x, y)$.

Make separate sketches of $S$ and $R$.

**SOLUTION:**

![Sketch of regions S and R](image)

(c) Find a translation $T$ where $T(S) = R$.

**SOLUTION:**

$T(u, v) = (u + 2, v + 1)$

(d) Use $T$ to reduce $\iint_R x \, dA$ to an integral over $S$, and then evaluate that new integral using polar coordinates.

**SOLUTION:**

The Jacobian of $T$ is just 1, as noted in part (a). So we have

$$ \iint_R x \, dA = \iint_S (u + 2) \, dA $$

Converting the second integral above to polar we have

$$ \iint_S (u + 2) \, dA = \int_0^{2\pi} \int_0^1 (r \cos \theta + 2) r \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{r^3 \cos \theta}{3} \right]_0^1 d\theta + 2\pi \left[ \frac{r^2}{2} \right]_0^1 $$
\[
\int_0^{2\pi} \cos \theta \, d\theta + 2\pi = 1/3 [\sin \theta]_0^{2\pi} + 2\pi = 2\pi
\]

3. Consider the region \( R \) shown below. Here the curved left side is given by \( x = y - y^2 \). In this problem, you will find a transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) which takes the unit square \( S = [0, 1] \times [0, 1] \) to \( R \).

![Image of the region R](image)

(a) As a warm up, find a transformation that takes \( S \) to the rectangle \([0, 2] \times [0, 1]\) which contains \( R \).

**SOLUTION:**

\[ L(u, v) = (2u, v) \]

(b) Returning to the problem of finding \( T \) taking \( S \) to \( R \), come up with formulas for \( T(u, 0) \), \( T(u, 1) \), \( T(0, v) \), and \( T(1, v) \). Hint: For three of these, use your answer in part (a).

**SOLUTION:**

\[ T(u, 0) = (2u, 0) \]
\[ T(u, 1) = (2u, 1) \]
\[ T(1, v) = (2, v) \]
\[ T(0, v) = (v - v^2, v) \]

(c) Now extend your answer in (b) to the needed transformation \( T \). Hint: Try “filling in” between \( T(0, v) \) and \( T(1, v) \) with a straight line.

**SOLUTION:**

\[ T(u, v) = (2u + v(1 - v)(1 - u), v) \]

(d) Compute the area of \( R \) in two ways, once using \( T \) to change coordinates and once directly.

**SOLUTION:**

To change coordinates we compute the Jacobian

\[
J(T) = \det \begin{pmatrix}
2 - v(1 - v) & (1 - 2v)(1 - u) \\
0 & 1
\end{pmatrix} = 2 - v(1 - v)
\]

So we have the area of \( R \) given by

\[
\iiint_R \, dxdy = \int_0^1 \int_0^1 2 - v(1 - v) \, du \, dv = 11/6
\]
Computing directly we have the area of $R$ given by
\[ \int_{0}^{1} 2 - (y - y^2) \, dy = \frac{11}{6} \]

4. If you get this far, evaluate the integrals in Problems 1 and 2 directly, without doing a change of coordinates. It's a fun-filled task…

**SOLUTION:**

For the integral in problem one, use the order $dy \, dx$. We need to split the double integral into three parts. The result is
\[
\iint_R x - 2y \, dA = \int_1^3 \int_{x/2}^{x} x - 2y \, dy \, dx + \int_1^{x/2+5/2} \int_{x/2}^{x} x - 2y \, dy \, dx + \int_2^{x/2+5/2} \int_{3x-5}^{x} x - 2y \, dy \, dx
\]
Evaluating this is not difficult but it is tedious. We leave it to the interested student. You should get $-\frac{25}{2}$.

For the integral in problem two, again use the order $dy \, dx$. We just need one double integral.
\[
\iint_R x \, dA = \int_1^3 \int_{1+\sqrt{1-(x-2)^2}}^{1-\sqrt{1-(x-2)^2}} x \, dy \, dx = \int_1^3 2x \sqrt{1-(x-2)^2} \, dx
\]
This integral can be evaluated by making the substitution $x - 2 = \sin u$, yielding the integral
\[
\int_{-\pi/2}^{\pi/2} (2 \sin u + 4) \cos^2 u \, du
\]
Now split this in two pieces as
\[
\int_{-\pi/2}^{\pi/2} 2 \sin u \cos^2 u \, du + \int_{-\pi/2}^{\pi/2} 4 \cos^2 u \, du
\]
The first is the integral of an odd function over an interval which is symmetric about the $y$ axis so it is 0. The second can be evaluated by using the trig identity $\cos^2 u = (1 + \cos 2u)/2$. This gives
\[
\int_{-\pi/2}^{\pi/2} 4 \cos^2 u \, du = \int_{-\pi/2}^{\pi/2} 4(1 + \cos 2u)/2 \, du = 2\pi.
\]