Purpose: In class, we've seen several different coordinate systems on \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) beyond the usual rectangular ones: polar, cylindrical, and spherical. The lectures on Friday and Monday will cover the crucial technique of simplifying hard integrals using a change of coordinates (Section 15.9). The point of this worksheet is to familiarize you with some basic concepts and examples for this process.

Starting point: Here we consider a variety of transformations \( T: \mathbb{R}^2 \to \mathbb{R}^2 \). Previously, we have used such functions to describe vector fields on the plane, but we can also use them to describe ways of distorting the plane:

1. Consider the transformation \( T(x, y) = (x - 2y, x + 2y) \).

   (a) Compute the image under \( T \) of each vertex in the below grid and make a careful plot of them, which should be fairly large as you will add to it later.

   To speed this up, divide the task up among all members of the group.

   \[
   \begin{array}{c|c|c|c|c|c}
   & y & & & & \\
   x & & & & & \\
   \hline
   0, 0 & & & & & \\
   2, 2 & & & & & \\
   2, -1 & & & & & \\
   -1, -1 & & & & & \\
   -1, 2 & & & & & \\
   \end{array}
   \]

SOLUTION:
See the image following part (f).
(b) For each pair $A$ and $B$ of vertices of the grid joined by a line, add the line segment joining $T(A)$ to $T(B)$ to your plot. This gives a rough picture of what $T$ is doing.

**SOLUTION:**

See the image following part (f).

(c) What is the image of the $x$-axis under $T$? The $y$-axis?

**SOLUTION:**

The image of the $x$-axis is the line $y = x$. The image of the $y$-axis is the line $y = -x$. To see this, parametrize the $x$-axis as $r(t) = (t, 0), -\infty < t < \infty$. Then $T(r(t)) = (t, t), -\infty < t < \infty$, which traces out the line $y = x$. Do the same for the $y$-axis.

(d) Consider the line $L$ given by $x + y = 1$. What is the image of $L$ under $T$? Is it a circle, an ellipse, a hyperbole, or something else?

**SOLUTION:**

Parametrize $L$ by $r(t) = (t, 1 - t), -\infty < t < \infty$. $T(L)$ is parametrized by $T(r(t)) = (t - 2(1 - t), t + 2(1 - t)) = (3t - 2, -t + 2)$. These are the parametric equations of a line.

(e) Consider the circle $C$ given by $x^2 + y^2 = 1$. What is the image of $C$ under $T$?

**SOLUTION:**

Parametrize $C$ by $r(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$. Then $T(r(t)) = (\cos t - 2\sin t, \cos t + 2\sin t), 0 \leq t \leq 2\pi$. Note that if we let $x = \cos t - 2\sin t, y = \cos t + 2\sin t$, then $y - x = 4\sin t$ and $y + x = 2\cos t$. So the curve $T(C)$ satisfies the equation $(\frac{y-x}{4})^2 + (\frac{y+x}{2})^2 = 1$. This is the equation of an ellipse.

(f) Add $T(L)$, $T(C)$ and $T(\text{your image})$ to your picture. Check your answer with the instructor.

**SOLUTION:**

![Image of points and lines](image)

**Note:** The transformation $T$ is a particularly simple sort called a *linear transformation*.

2. Consider the transformation $T(x, y) = (y, x(1 + y^2))$. Draw the image of the picture below under $T$. 
SOLUTION:

Label the 5 line segments as at left below. The image of the left hand picture is the right hand picture.

We can figure this out as follows. First parametrize the line segments:

\[ \mathbf{r}_A(t) = (0, t), 0 \leq t \leq 1 \]
\[ \mathbf{r}_B(t) = (t, 0), 0 \leq t \leq 1 \]
\[ \mathbf{r}_C(t) = (1, t), 0 \leq t \leq 1 \]
\[ \mathbf{r}_D(t) = (t, 1), 0 \leq t \leq 1 \]
\[ \mathbf{r}_E(t) = (t, t), 0 \leq t \leq 1 \]

Then compute the image under \( T \) of each of these:

\[ T(\mathbf{r}_A(t)) = (t, 0), 0 \leq t \leq 1 \]
\[ T(\mathbf{r}_B(t)) = (0, t), 0 \leq t \leq 1 \]
\[ T(\mathbf{r}_C(t)) = (t, 1 + t^2), 0 \leq t \leq 1 \]
\[ T(\mathbf{r}_D(t)) = (1, 2t), 0 \leq t \leq 1 \]
\[ T(\mathbf{r}_E(t)) = (t, t(1 + t^2)), 0 \leq t \leq 1 \]

Graphing each of these gives the image above at left.

3. In this problem, you’ll construct a transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) which rotates counter-clockwise about the origin by \( \pi/4 \), as shown below.
(a) Give a formula for $T$ in terms of polar coordinates. That is, how does rotation affect $r$ and $\theta$?

**SOLUTION:**

$$T(r, \theta) = (r, \theta + \pi/4)$$

(b) Write down $T$ in terms of the usual rectangular $(x, y)$ coordinates. Hint: first convert into polar, apply part (a) and then convert back into rectangular coordinates.

**SOLUTION:**

First convert $(x, y)$ into polar:

$$(r, \theta) = (\sqrt{x^2 + y^2}, \arctan(y/x))$$

Then apply $T$ in polar coordinates:

$$T(r, \theta) = (r, \theta + \pi/4)$$

Then convert the result to rectangular coordinates:

$$T(x, y) = (r \cos(\theta + \pi/4), r \sin(\theta + \pi/4)),$$

where $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$.

Recall the double angle formulas $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$. Using these we see that

$$\cos(\theta + \pi/4) = \cos \theta \cos(\pi/4) - \sin \theta \sin(\pi/4) = \sqrt{2}/2(\cos \theta - \sin \theta)$$

and

$$\sin(\theta + \pi/4) = \sin \theta \cos(\pi/4) + \sin(\pi/4) \cos \theta = \sqrt{2}/2(\sin \theta + \cos \theta).$$

Hence we have

$$r \cos(\theta + \pi/4) = \sqrt{2}/2(r \cos \theta - r \sin \theta) = \sqrt{2}/2(x - y)$$

and

$$r \sin(\theta + \pi/4) = \sqrt{2}/2(r \sin \theta + r \cos \theta) = \sqrt{2}/2(x - y).$$

So we have

$$T(x, y) = (\sqrt{2}/2(x - y), \sqrt{2}/2(x - y)).$$