Thursday, October 11  *  Solutions  *  Curves and integration.

1. Consider the curve $C$ in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = (e^t \cos t) \mathbf{i} + 2 \mathbf{j} + (e^t \sin t) \mathbf{k}$$

(a) Draw a sketch of $C$.

**Solution.** The sketch of $C$ is the following graph.

![Figure 1: Sketch of $C$.](image)

(b) Calculate the arc length function $s(t)$, which gives the length of the segment of $C$ between $\mathbf{r}(0)$ and $\mathbf{r}(t)$ as a function of the time $t$ for all $t \geq 0$. Check your answer with the instructor.

**Solution.** Since

$$x'(t) = e^t \cos t - e^t \sin t, \quad y'(t) = 0, \quad z'(t) = e^t \sin t + e^t \cos t,$$

we have

$$|\mathbf{r}'(t)| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2} e^t.$$

Hence the arc length is

$$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2} e^u \,du = \sqrt{2} e^t - \sqrt{2}.$$
(c) Now invert this function to find the inverse function \( t(s) \). This gives time as a function of arclength, that is, tells how long you must travel to go a certain distance.

**Solution.** Solve \( s = \sqrt{2}e^t - \sqrt{2} \), which gives \( e^t = \frac{s + \sqrt{2}}{\sqrt{2}} \), and so

\[
t = t(s) = \ln \left( \frac{s + \sqrt{2}}{\sqrt{2}} \right).
\]

(d) Suppose \( h : \mathbb{R} \to \mathbb{R} \) is a function. We can get another parameterization of \( C \) by considering the composition

\[
f(s) = r(h(s))
\]

This is called a reparametrization. Find a choice of \( h \) so that

i. \( f(0) = r(0) \)

ii. The length of the segment of \( C \) between \( f(0) \) and \( f(s) \) is \( s \). (This is called parametrizing by arc length.)

Check your answer with the instructor.

**Solution.** From (c) we know \( t = \ln \left( \frac{s + \sqrt{2}}{\sqrt{2}} \right) \). When \( s = 0 \), we have \( t = \ln 1 = 0 \). Then we can choose

\[
h(s) = \ln \left( \frac{s + \sqrt{2}}{\sqrt{2}} \right).
\]

(e) Without calculating anything, what is \( |f'(s)| \)?

**Solution.** Since \( s = \int_0^s |f'(u)| \, du \), then by the fundamental theorem of calculus, we can differentiate both sides with respect to \( s \) and get \( 1 = |f'(s)| \).

2. Consider the curve \( C \) given by the parametrization \( r : \mathbb{R} \to \mathbb{R}^3 \) where \( r(t) = (\sin t, \cos t, \sin^2 t) \).

(a) Show that \( C \) is in the intersection of the surfaces \( z = x^2 \) and \( x^2 + y^2 = 1 \).

**Solution.** Since \( x = \sin t, y = \cos t, z = \sin^2 t \), it is very easy to check that \( z = x^2 \) and \( x^2 + y^2 = 1 \). So the curve \( C \) lies in both these two surfaces, hence is in the intersection of them.

(b) Use (a) to help you sketch the curve \( C \).

**Solution.** The left graph is the intersection of the two surfaces, while the right one is the curve.
3. (a) Sketch the top half of the sphere \(x^2 + y^2 + z^2 = 5\). Check that \(P = (1, 1, \sqrt{3})\) is on this sphere and add this point to your picture.

**Solution.** The top half of the sphere is shown in Figure 3 (the black dot is \(P\)). Since \(1^2 + 1^2 + (\sqrt{3})^2 = 5\), we know \(P\) is on this sphere.

![Figure 3: Half sphere and the path.](image)

(b) Find a function \(f(x, y)\) whose graph is the top-half of the sphere. Hint: solve for \(z\).

**Solution.** Since \(x^2 + y^2 + z^2 = 5\), we have \(z^2 = 5 - x^2 - y^2\), and so \(z = \pm \sqrt{5 - x^2 - y^2}\). As we only want the top half of the sphere, we can let \(f(x, y) = \sqrt{5 - x^2 - y^2}\).

(c) Imagine an ant walking along the surface of the sphere. It walks *down* the sphere along...
the path $C$ that passes through the point $P$ in the direction parallel to the $yz$-plane. Draw this path in your picture.

**Solution.** The black curve in Figure 3 is the path.

(d) Find a parametrization $\mathbf{r}(t)$ of the ant’s path along the portion of the sphere shown in your picture. Specify the domain for $\mathbf{r}$, i.e. the initial time when the ant is at $P$ and the final time when it hits the $xy$-plane.

**Solution.** $x = 1$ along the path and $f(1, y) = \sqrt{4 - y^2}$, so setting $y = t$ we get the parametrization

$$\mathbf{r}(t) = (1, t, \sqrt{4 - t^2}).$$

4. As in 1(d), consider a reparametrization

$$\mathbf{f}(s) = \mathbf{r}(h(s))$$

of an arbitrary vector-valued function $\mathbf{r}: \mathbb{R} \to \mathbb{R}^3$. Use the chain rule to calculate $|\mathbf{f}'(s)|$ in terms of $\mathbf{r}'$ and $h'$.

**Solution.** By the chain rule, $\mathbf{f}'(s) = \mathbf{r}'(h(s)) \cdot h'(s)$. Taking magnitudes of both sides we have

$$|\mathbf{f}'(s)| = |\mathbf{r}'(h(s))| \cdot |h'(s)|.$$