1. Let $R$ be the region of integration pictured below right. Evaluate $\iint_R 6y \, dA$. (4 points)

\[ \int_{-y}^{y+1} \int_0^1 6y \, dx \, dy \]

\[ = \int_0^1 6y \cdot \left[ x \right]_x=-y \, dy = \int_0^1 6y (2y+1) \, dy \]

\[ = \int_0^1 12y^2 + 6y \, dy = 4y^3 + 3y^2 \bigg|_{y=0}^{y=1} \]

\[ = 7 \]

\[ \iint_R 6y \, dA = 7 \]

2. Set up, but DO NOT EVALUATE, a triple integral that computes the volume of the tetrahedron shown at right. (5 points)

Two common correct answers:

\[ \int_0^1 \int_0^{1-x} \int_0^{1-x} 1 \, dz \, dy \, dx \]

\[ \int_0^1 \int_0^{1-y} \int_0^{1-y} 1 \, dz \, dx \, dy \]

3. Set up, but DO NOT EVALUATE, a triple integral that computes the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$. (5 points)

Cross section:

3D:

Using spherical coordinates.

\[ \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^{\sqrt{2}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

\[ = 1 \, dV \]
4. Consider the triple integral \( \int_0^{1/2} \int_0^{1-y} \int_0^{1-x^2-y^2} f(x, y, z) \, dz \, dx \, dy \). Mark the corresponding region of integration below. **(3 points)**

The projection on xy plane is \( D = \{(x, y) : 0 < x < \frac{1}{2}, y < x^2 + y \} \). So the only possible solutions are the circled ones.

\[ x = 1 - y \]
\[ z = 1 - (1-y)^2 - y^2 = 1 - (1-y)(1-y+y) = 2y \]

But when \( x = 1 - y \), then \( z = 1 - (1-y)^2 - y^2 = 1 - (1-y)(1-y+y) = 2y \) while in the right circled one \( z = 0 \) when \( x = 1 - y \). So it is not the right circled graph.

5. Find a transformation \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which takes the unit circle to the ellipse given by \( (x - 3)^2 + \frac{y^2}{4} = 1 \) as shown. **(3 points)**

\( T(u, v) = (u+3, 2v) \)
6. Let \( R \) be the region in the \( xy \)-plane depicted below right. Let \( T(u, v) = (2u + v, u - v) \).

(a) Find a rectangle \( S \) in the \( uv \)-plane whose image under \( T \) (that is, the collection of points \( T(u, v) \) for all choices of \((u, v)\) in \( S \)) is exactly \( R \). \( \text{(3 points)} \)

\[
T(1,0) = (2,1) \\
T(0,1) = (1,-1)
\]

So \( 0 \leq u \leq 1 \).

and to get image down to \((2,-2)\) want:

\[
\text{ANSWER: } S = \{(u,v) \mid 0 \leq u \leq 1, \ 0 \leq v \leq 2\}.
\]

(b) Set up, but DO NOT EVALUATE, the integral \( \iint_R \cos(x) \, dA \) as an integral in the \((u,v)\)-coordinates. If you can't do part (a), leave the limits of integration blank. \( \text{(5 points)} \)

\[
\iint_R \cos(x) \, dA = \int_0^2 \int_0^1 \cos(2u + v) \, | \det J | \, du \, dv
\]

\[
J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}
\]

\[
| \det J | = 2(-1) - 1 = -3
\]

\[
\iint_R \cos(x) \, dA = \int_0^2 \int_0^1 3 \cos(2u + v) \, du \, dv
\]

7. Let \( F(x,y) = (x^2, x^2 \cos(y)) \). Then \( \iint_R \left[ \frac{\partial}{\partial x} (x^2 \cos(y)) - \frac{\partial}{\partial y} (x^2) \right] \, dA = 0 \) where \( R \) is the region shown below.

Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the pictured curve that goes from \((-3,0)\) to \((3,0)\) via \((0,2)\). \( \text{(3 points)} \)

By Green, we have \( \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = 0 \)

Thus \( \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \)

\[
= \int_C x^2 \, dx + \int_C x^2 \cos(y) \, dy
\]

\[
= \int_{-3}^3 x^2 \, dx = \left. \frac{x^3}{3} \right|_{-3}^3 = 18
\]

\( C' = \text{bottom of } R \).
8. Consider the region $D$ in the plane bounded by the curve $C$ as shown at right. For each part, circle the best answer. (1 point each)

(a) For $F(x, y) = (x + 1, y^2)$, the integral $\int_C F \cdot dr$ is

- negative
- zero
- positive

(b) The integral $\int_C (-y\, dx + 2\, dy)$ is

- negative
- zero
- positive

(c) The integral $\iint_D (y-x)\, dA$ is

- negative
- zero
- positive

Split $D$ into $D_1$ and $D_2$.

Then $\iint_D (y-x)\, dA = \iint_{D_1} (y-x)\, dA + \iint_{D_2} (y-x)\, dA$

and $\iint_{D_2} (y-x)\, dA > 0$ and $\iint_{D_1} (y-x)\, dA > 0$,

so $\iint_D (y-x)\, dA$ has to be negative.

9. For each surface $S$ below, give a parameterization $r : D \to S$. Be sure to explicitly specify the domain $D$ and call your parameters $u$ and $v$.

(a) The rectangle in $\mathbb{R}^3$ with vertices $(1,0,0), (0,1,0), (1,0,2), (0,1,2)$. (3 points)

Can use coordinates $x$ and $z$ as parameters, call them $u$ and $v$. Since this rect. is contained in the line $x+y=1$, get $y=1-x=1-u$

$D = \{ 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 2 \}$

$r(u,v) = \langle u, 1-u, v \rangle$

(b) The portion of cone $y = \sqrt{x^2 + z^2}$ for $0 \leq y \leq 1$ which is shown at right. (4 points)

Can use $v = y$ and the angle $u$ shown as the parameters. Also, the radius of the circle with $y$ fixed is just $y$,

and hence:

$D = \{ 0 \leq u \leq 2\pi \text{ and } 0 \leq v \leq 1 \}$

$r(u,v) = \langle v \cos u, v, v \sin u \rangle$
10. Consider the surface $S$ parameterized by $r(u, v) = (v^2, u, v)$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

(a) Mark the correct picture of $S$ below. (2 points)

Along $x = 0$ we get $v^2 = 0 \Rightarrow v = 0$ so the path on the surface will be $(0, y, 0) \in \text{cy-axis}$.

So we are between the circled ones.

But the $x$-coordinate of the surface is $v^2 \geq 0$, so it can't be the right circled one.

(b) Evaluate the integral $\iiint_S z \, dA$. (6 points)

$\iiint_S z \, dA = \int_0^1 \int_0^1 \int_0^1 \sqrt{1 + 4v^2} \, du \, dv = \int_0^1 \int_0^1 \sqrt{1 + 4v^2} \, du \, dv$

$= \int_0^1 \sqrt{1 + 4v^2} \, dv = \int_0^1 \frac{1}{2} \sqrt{w} \, dw$

$= \frac{1}{12} \omega^{3/2} \bigg|_{w=0}^{w=1} = \frac{1}{12} (5^{3/2} - 1)$

$\iiint_S z \, dA = \frac{1}{12} (5^{3/2} - 1)$

11. Consider the solid described as follows using cylindrical coordinates: $E$ is the region inside the paraboloid $z = 1 - r^2$ and where $0 \leq \theta \leq \pi$ and $z \geq 0$. Choose one double integral and one triple integral below that compute the volume of $E$. (1 point each)

- $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dz \, dx$
- $\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{1-z^2}} 1 \, dy \, dz$
- $\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{1-z^2}} 1 \, dy \, dz$
- $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dz \, dx$
- $\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{1-y^2} \, dy \, dz$
- $\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{1-y^2-z^2} \, dy \, dz$
- $\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{1-y^2-z^2} \, dy \, dz$
- $\int_0^1 \int_{\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{(1-z)^2-y^2} \, dy \, dz$

On the $xy$-plane, $y = \text{sign}(z) \times \sqrt{1-z^2}$, so the two first integrals are 0. We get by the green lines: $z = 1 - r^2$, $z = 1 - r^2$, $z = 1 - r^2$. The angle of $E$ with the $x$-axis, $\frac{\pi}{2}$. The interior of $E$ is the angle below $z = 1 - r^2$. We get by the green lines: $z = 1 - r^2$, $z = 1 - r^2$, $z = 1 - r^2$. The angle of $E$ with the $x$-axis, $\frac{\pi}{2}$.