1. Find the maximum value of the function \( f(x, y) = 3x + y \) on the curve \( x^2 + y^2 = 10 \). (5 points)

We use Lagrange multipliers where the constraint is

\[ g = 10 \text{ for } g = x^2 + y^2. \]  

The critical pts are where

\[ \nabla f = \langle 3, 1 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle \]

and hence we solve:

\[
\begin{align*}
3 &= \lambda \cdot 2x \\
1 &= \lambda \cdot 2y \\
x^2 + y^2 &= 10
\end{align*}
\]

Clearly, \( \lambda \neq 0 \) as else we get \( 3 = 0 \), so we solve for \( 1/\lambda \) to find

\[
\frac{1}{\lambda} = \frac{2}{3} \Rightarrow x = 2y \Rightarrow x = 3y.
\]

Then \( x^2 + y^2 = (3y)^2 + y^2 = 10y^2 = 10 \) and so \( y = \pm 1 \). Thus the critical points are \((3,1)\) and \((-3,-1)\). As a circle is closed and bounded, the Extreme Value Theorem tells us we need only check the values of \( f \) at these two points (get 10 and -10) to learn that:

\[
\text{Max. value} = 10
\]
2. The table below contains data about a differentiable function \( g(x, y) \) at several points. Find all the critical points from among the points listed below, and determine whether each is a local maximum, local minimum, or saddle point.

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(g(x, y))</th>
<th>(g_x(x, y))</th>
<th>(g_y(x, y))</th>
<th>(g_{xx}(x, y))</th>
<th>(g_{yy}(x, y))</th>
<th>(g_{xy}(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-2, 0))</td>
<td>-16</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>((0, 0))</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>((2, 0))</td>
<td>16</td>
<td>0</td>
<td>0</td>
<td>-12</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\begin{vmatrix} 12 & 0 \\ 0 & 2 \end{vmatrix} = 24 > 0
\]
\[
\begin{vmatrix} -12 & 0 \\ 0 & 2 \end{vmatrix} = -24 < 0
\]

For each of these points, circle the phrase that makes the sentence true. (1 point each)

<table>
<thead>
<tr>
<th>((-2, 0)) is</th>
<th>not a critical point</th>
<th>a local minimum</th>
<th>a local maximum</th>
<th>a saddle point</th>
<th>of ( g ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0)) is</td>
<td>not a critical point</td>
<td>a local minimum</td>
<td>a local maximum</td>
<td>a saddle point</td>
<td>of ( g ).</td>
</tr>
<tr>
<td>((2, 0)) is</td>
<td>not a critical point</td>
<td>a local minimum</td>
<td>a local maximum</td>
<td>a saddle point</td>
<td>of ( g ).</td>
</tr>
</tbody>
</table>

3. The vector field \( \mathbf{F} = \langle y^2, 2xy + 3y^2 \rangle \) is conservative. Find a function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( \mathbf{F} = \nabla f \). \textbf{Note:} Check your answer—no partial credit will be given on this problem. (3 points)

\[
f(x, y) = xy^2 + y^3
\]

---

**Scratch space below**

Want \( f(x, y) \) with \( \frac{\partial f}{\partial x} = y^2 \), so integrating gives

\[ f = xy^2 + C(y) \]

Computing \( \frac{\partial f}{\partial y} = 2xy + \frac{\partial C}{\partial y} \)

Thus we need \( \frac{\partial C}{\partial y} = 3y^2 \), so can take \( C = y^3 \).

Hence \( f = xy^2 + y^3 \).
4. The contour plot of a differentiable function $f$ is shown below. (2 points each)

Length of $C$ is about 1.75 = $7/4$. To estimate the average on $C$, break into 4 equal pieces shown, each with averages 
$[-10, -6, -3, -2]$ to estimate the average as $-21/4$. So $\int_C f \approx \left(-\frac{21}{4}\right)\left(\frac{7}{4}\right) \approx -9.2$

(a) Circle the phrase that makes this sentence true: “The point $A$ is ______ of $f$.”

- not a critical point
- a local maximum
- a local minimum
- a saddle point

(b) Circle the statement which best describes the relationship between the directional derivatives $D_uf(P)$ and $D_uf(Q)$, where $u$ is the unit vector indicated at the points $P$ and $Q$.

- $D_uf(P) > D_uf(Q)$
- $D_uf(P) < D_uf(Q)$
- $D_uf(P) = D_uf(Q)$

(c) For $C$ the oriented curve shown above, evaluate the line integral $\int_C \nabla f \cdot dr$.

-12 -9 -6 -3 0 3 6 9 12

(d) Estimate the value of $\int_C f(x, y) \, ds$:

-12 -9 -6 -3 0 3 6 9 12 (see above)

5. Which one of the following vector fields is conservative? Circle your answer. In each case the domain is the set of points where the formula makes sense. (2 points)

\[
\left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle, \quad (x^2, y^2), \quad (y^2, x^2), \quad (xy, xy)
\]
6. Let $f(x, y)$ be a differentiable function on the disk $D = \{x^2 + y^2 \leq 100\}$ in $\mathbb{R}^2$, where

- $f(x, y) = 20$ for every point $(x, y)$ on the circle $x^2 + y^2 = 100$.
- $f(0, 0) = 5$.
- $f$ has only one critical point, which is at $(1, 2)$.

Decide which of the four statements below is true and mark the box next to it. (2 points)

- $f(1, 2) > 5$
- $\sqrt{f(1, 2) < 5}$$
- $f(1, 2) = 5$
- The relationship between $f(1, 2)$ and 5 cannot be determined from the given information.

7. Consider the following four regions in the plane: (1 point each)

- $R_1 = \{1 < x^2 + y^2 < 4\}$
- $R_2 = \{y \geq 0\}$
- $R_3 = \{1 < x^2 + y^2 < 4 \text{ and } y \geq 0\}$
- $R_4 = \{1 < x^2 + y^2 < 4 \text{ and } y < 0\}$

(a) Which region is neither open nor closed?

(b) Which region is not simply connected?

8. Mark the box below the only one of these three vector fields which is not conservative. (2 points)

Integrals on vertical sides cancel each other out, int. along the top is $> 0$ and bottom is $< 0$.
9. Find a vector function \( \mathbf{r}(t) \) that represents the curve of intersection of the cylinder \( y^2 + z^2 = 4 \) and the hyperbolic paraboloid \( x = z^2 - y^2 \). Specify the range of the parameter \( t \) so that your answer traces the curve exactly once. \( \text{(4 points)} \)

Can parametrize a circle of radius 2 in the \((y, z)\) plane by
\[
y = 2 \cos t, \quad z = 2 \sin t.
\]
Now as \( x = z^2 - y^2 \) we get
\[
\mathbf{r}(t) = \left( \frac{4 \sin^2 t}{4 \cos^2 t}, 2 \cos t, 2 \sin t \right) \quad \text{for} \; 0 \leq t \leq 2\pi.
\]

10. Let \( C \) be the oriented curve parameterized by \( \mathbf{r}(t) = (\cos t, \sin t, e^t), 0 \leq t \leq 8\pi. \)

(a) Check the box next to the picture which best matches \( C \). \( \text{(2 points)} \)

(b) Calculate the line integral \( \int_C z \, dz \). \( \text{(4 points)} \)

Here \( z = e^t \) so \( dz = z'(t) \, dt = e^t \, dt \).

Hence
\[
\int_C z \, dz = \int_0^{8\pi} e^t \cdot e^t \, dt = \int_0^{8\pi} e^{2t} \, dt
\]
\[
= \frac{1}{2} e^{2t} \bigg|_{t=0}^{8\pi} = \frac{1}{2} (e^{16\pi} - 1)
\]

\[
\int_C z \, dz = \frac{1}{2} (e^{16\pi} - 1)
\]
11. Consider the surface $S$ defined by the equation $x^3 + y^3 + z^3 = -8$. Find an equation for the plane tangent to $S$ at the point $(1, -1, -2)$. (4 points)

Setting $g = x^3 + y^3 + z^3$, we know a normal vector to $S$ at $(1, -1, -2)$ is given by

$$\nabla g(1, -1, -2) = \langle 3, 3, 12 \rangle$$

since $\nabla g = \langle 3x^2, 3y^2, 3z^2 \rangle$. Taking $\vec{n} = \langle 1, 1, 4 \rangle$
we get:

$$1 \cdot (x - 1) + 1 \cdot (y + 1) + 4 \cdot (z + 2) = 0$$

which gives

Equation: $1 \boxed{x} + 1 \boxed{y} + 4 \boxed{z} = -8$

12. Find the mass of a thin wire in the shape of the curve $x = \sin t$, $y = 2 \sin t$, $z = \sqrt{5} \cos t$, $0 \leq t \leq \pi$, if the wire has density function $\rho(x, y, z) = x$. (5 points)

$$\text{mass} = \int_C \rho \, ds = \int_0^\pi (\sin t) \sqrt{5} \, dt = \sqrt{5} \cos t \bigg|_0^\pi = \sqrt{5} \cdot (1 - (-1))$$

$$\vec{r}'(t) = \langle \cos t, 2 \cos t, -\sqrt{5} \sin t \rangle$$

$$|\vec{r}'(t)| = \sqrt{\cos^2 t + 4 \cos^2 t + 5 \sin^2 t} = \sqrt{5}$$

Total mass = $2\sqrt{5}$