1. Let \( \mathbf{v}, \mathbf{a}, \) and \( \mathbf{b} \) be the vectors (in the plane of the paper) drawn at right, all of which have length 1. Let \( \mathbf{w} \) be a vector of length two pointing directly out of the paper. Which of the following vectors is \( \mathbf{v} \times \mathbf{w} \)? (2 points)

\[
\begin{array}{cccccccc}
-2\mathbf{b} & -\mathbf{b} & \mathbf{b} & 2\mathbf{b} & (0,0,0) & -2\mathbf{a} & -\mathbf{a} & \mathbf{a} & 2\mathbf{a}
\end{array}
\]

2. A function \( u: \mathbb{R}^2 \to \mathbb{R} \) is harmonic if it satisfies Laplace's equation: \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \). Check the box next to the unique graph below that corresponds to a harmonic function. (2 points)

3. Let \( f \) be a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \). Suppose that \( f(x, y) \to 3 \) as \((x, y)\) approaches \((0,1)\) along every line of the form \( y = kx + 1 \). What can you say about the limit \( \lim_{(x,y)\to(0,1)} f(x, y) \)? Check the box next to the correct statement. (2 points)

- It exists and is equal to 3.
- We cannot determine if the limit exists, but if it does, the limit is 3.
- It does not exist.

4. Suppose we know the following data about \( g: \mathbb{R}^3 \to \mathbb{R} \).

<table>
<thead>
<tr>
<th>((x, y, z))</th>
<th>(g(x, y, z))</th>
<th>(g_x(x, y, z))</th>
<th>(g_y(x, y, z))</th>
<th>(g_z(x, y, z))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3,3,1))</td>
<td>30</td>
<td>6</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>((0.5,0.1,0))</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Circle the best estimate for \( g(3.5,3.1,1) \): 30.6, 31.0, 32.6, 33.4, 34.1 (2 points)

\[
g(3.5,3.1,1) \approx g(3,3,1) + g_x(3,3,1) \Delta x + g_y(3,3,1) \Delta y
\]

\[
= 30 + 6 \cdot (0.5) + 4 \cdot (0.1) = 30 + 3 + 0.4 = 33.4
\]
5. Let $S$ be the surface $x^2 + y^2 = -z^2$ shown at right. Let $f$ be a function on $\mathbb{R}^3$ with continuous partial derivatives such that

\[
\nabla f(0,0,0) = (1,1,3) \quad \nabla f(2,2,-2) = (1,1,3) \\
\nabla f(-2,-2,-2) = (1,1,3) \quad \nabla f(2,-2,2) = (0,0,0)
\]

Circle every point below that can not be the point at which $f$ achieves its minimum value on $S$. (4 points)

(0,0,0)  (2,2,-2)  (-2,-2,-2)  (2,-2,-2)

6. Let $S$ be the surface parametrized by $r(u,v) = (v \cos u, u, v)$ for $-\pi/2 \leq u \leq \pi/2$ and $-1 \leq v \leq 1$. Check the box next to the picture of $S$ below: (2 points)

7. Let $F(x,y,z) = (e^x, 3x + 2xye^y)$, and let $C$ be the oriented curve at right. Estimate the value of $\int_C F \cdot dr$. (2 points)

-9 -6 -3 0 3 6 9

---

**Scratch Space**

$5. \quad \text{Set } g = x^2 + y^2 + z^2 \text{ and then the places where the min could occur is when } \nabla f = \lambda \nabla g \text{ or } \nabla g = \delta.$

$7. \quad \int_C F \cdot dr = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 3 \text{ Area}(\square) \approx 3.$
8. A vector field \( \mathbf{F} \) is shown at right; for scale, here \( \mathbf{F}(0,0) = (0, 0.1) \). Assuming that \( \mathbf{F} \) is conservative, circle the value of \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the curve shown from \((0, -1)\) to \((0, 1)\).

\[
\begin{array}{cccccc}
-0.3 & -0.2 & -0.1 & 0 & 0.1 & 0.2 & 0.3
\end{array}
\]

(2 points)

Same as integral along the portion of the y-axis from \((0, -1)\) to \((0, 1)\), which is \((0.1 \times 2) = 0.2\).

9. Consider the transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by \( T(u, v) = (2u - v, 2u + v) \). Let \( P \) be the rectangle shown below in the \((x, y)\)-plane, drawn against a unit-square grid. Check the box next to the region \( D \) in the \((u, v)\)-plane below that is mapped to \( P \) by \( T \). (2 points)

- \( D = \{0 \leq u \leq 1 \text{ and } 0 \leq v \leq 1\} \)
- \( D = \{0 \leq u \leq 1 \text{ and } 0 \leq v \leq 4\} \)
- \( D = \{-1 \leq u \leq 0 \text{ and } 0 \leq v \leq 1\} \)
- \( D = \{0 \leq u \leq 2 \text{ and } 0 \leq v \leq 1\} \)
- \( D = \{0 \leq u \leq 4 \text{ and } 0 \leq v \leq 1\} \)
- \( D = \{0 \leq u \leq 1 \text{ and } 0 \leq v \leq 2\} \)

\[
\begin{array}{c}
T(1, 0) = (2, 2) \\
T(0, 1) = (-1, 1)
\end{array}
\]

Scratch Space
10. Consider the curve \( C \) parametrized by \( \mathbf{r}(t) = (\cos t, \sin t, \cos 2t) \), where \( 0 \leq t \leq 2\pi \).

(a) Check the box next to the correct sketch of \( C \). \( \textbf{(2 points)} \)

(b) Find the work done if a particle travels along path \( \mathbf{r}(t) \) under the force field given by \( \mathbf{F}(x, y, z) = (-2y, 2x, 0) \). \( \textbf{(4 points)} \)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ -2\sin 2t \end{pmatrix} \, dt
\]

\[
= \int_0^{2\pi} 2(\sin^2 t + \cos^2 t) \, dt
\]

\[
= 4\pi
\]

Total work done = \( 4\pi \)

11. Let \( S = \{ u^2 + v^2 + w^2 = 1 \} \) be the unit sphere around the origin, and let \( E = \{ 4x^2 + (y+1)^2 + (z-3)^2 = 1 \} \), which is an ellipsoid with center \((0, 1, 3)\). Find a transformation \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) such that \( T(S) = E \). \( \textbf{(3 points)} \)

\[ T(u, v, w) = \left( \frac{u}{2}, v + 1, w + 3 \right) \]

Need to scale 1st coor down by \( \frac{1}{2} \), then shift in the other two coor.

12. Find the volume of the solid that lies below the cone \( z = -\sqrt{x^2 + y^2} \) and inside the sphere \( x^2 + y^2 + z^2 = 4 \). (5 points)

\[
\text{Vol}(E) = \int_0^2 \int_0^{\pi / 4} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho
\]

\[
= \int_0^2 \rho^2 \, d\rho \int_0^{2\pi} d\theta \int_0^{\pi / 4} \sin \phi \, d\phi
\]

\[
= \left( \frac{\rho^3}{3} \right) \cdot 2\pi \cdot \left( -\cos \phi \right) \bigg|_0^{\pi / 4}
\]

\[
= \frac{8}{3} \cdot 2\pi \left( 1 - \frac{1}{\sqrt{2}} \right)
\]

Write your volume integral here:

\[
\int_0^2 \int_0^{\pi / 4} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho
\]

Your final answer: Volume = \( \frac{16\pi}{3} \left( 1 - \frac{1}{\sqrt{2}} \right) \)
13. For each of the integrals: (A) $\int_0^\pi \int_0^1 \int_{1-r^2}^1 f(r, \theta, z) r \, dz \, dr \, d\theta$ and (B) $\int_0^\pi \int_0^1 \int_0^{1-z} f(r, \theta, z) r \, dr \, dz \, d\theta$

label the solid corresponding to the region of integration below. (2 points each)

![Solid Integrals](image)

14. Label the boxes next to the parametrizations that correspond to the following two surfaces: (2 points each)

- **B** $r(u, v) = \left< \frac{\cos u \, \sin v}{1 + u^2}, \frac{\sin u}{1 + u^2}, u \right>$ for $-1 \leq u \leq 1$, $0 \leq v \leq 2\pi$

- **A** $r(u, v) = \left< \cos v, \sin u, v \right>$ for $-1 \leq u \leq 1$, $0 \leq v \leq 2\pi$

- **A** $r(u, v) = \left< \cos v, u \sin v, 1 - u^2 \right>$ for $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$

- $r(u, v) = \left< \sin u \cos v, \sin u \sin v, \cos u \right>$ for $0 \leq u \leq \pi/2$, $0 \leq v \leq 2\pi$
15. Here are plots of five vector fields on the box where \(0 \leq x \leq 1\), \(0 \leq y \leq 1\), and \(0 \leq z \leq 1\). (2 points each)

(a) Circle the name of the vector field that is given by \((-z, 0, 1+x)\): A B C D E

(b) Exactly one of these vector fields has nonzero divergence. Circle it: A B C D E

(c) The vector field C is conservative: true false

(d) Which vector field is the gradient of a function \(f\) whose level sets are shown below? A B C D E
16. Let $E$ be the solid region shown below, where $\partial E$ is decomposed into the four subsurfaces $S_i$ indicated; here the top $S_0$ is where $z + x^2 = 1$, the front $S_1$ is in the $xz$-plane, the back is $S_2$, and the bottom is $S_3$.

(a) Use a triple integral to compute the volume of $E$. \( 4 \) points

\[
\int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{1} \, dz \, dy \, dx = \int_{-1}^{1} \left( 1 - x^2 \right) dx = 2 - 2x^2 \left|_{x=-1}^{1} \right. = \frac{8}{3}
\]

Vol($E$) = $\frac{8}{3}$

(b) Give a parameterization of $S_0$ and use it to directly compute the flux of $\mathbf{F} = \langle 1, 0, z + 2 \rangle$ through $S_0$ with respect to the upwards normals. \( 5 \) points

$\mathbf{r}(u,v) = \langle u, v, 1 - u^2 \rangle$  \( D = \{ -1 \leq u \leq 1 \text{ and } 0 \leq v \leq 2 \} \)

$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & 0 \end{vmatrix} = \langle 2u, 0, 1 \rangle$

$\mathbf{F}(\mathbf{r}(u,v)) = \langle 1, 0, 3 - u^2 \rangle$

$\mathbf{\text{Flux}} = \int_{0}^{2} \int_{-1}^{1} \mathbf{F}(\mathbf{r}(u,v)) \cdot (2u, 0, 1) \, du \, dv$

$= \int_{0}^{2} \int_{-1}^{1} 2u + 3 - u^2 \, du \, dv$

$= \int_{0}^{2} \left( \frac{u^2}{2} + 3u - \frac{u^3}{3} \right)_{u=-1}^{1} \, dv$

$= \int_{0}^{2} \left( \frac{8}{3} \right) \, dv$

$= \frac{16}{3}$

\[ \iint_{S_0} \mathbf{F} \cdot d\mathbf{S} = \frac{32}{3} \]

(c) The flux of $\mathbf{F}$ through exactly two of $S_1$, $S_2$, and $S_3$ is zero. Circle the one where the flux is nonzero. \( 1 \) point

\[ S_1 \quad S_2 \quad \boxed{S_3} \]
17. Consider the vector field \( \mathbf{F} = (-y, x+z, x^2+z) \) on \( \mathbb{R}^3 \).

(a) Circle the curl of \( \mathbf{F} \): (2 points)

\[
\text{curl} \mathbf{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\end{vmatrix}
= \langle -1, -2x, 2 \rangle
\]

(b) Suppose \( C \) is a closed curve in the plane \( P \) given by \( x - z = 1 \). Assuming \( C \) bounds a region \( R \) of area 10 in \( P \), determine the absolute value of \( \int_C \mathbf{F} \cdot d\mathbf{r} \). (4 points)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\text{curl} \mathbf{F}) \cdot \hat{n} \, dA = \iint_R \langle -1, -2x, 2 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle \, dA
\]

\[
\iint_R \frac{1}{\sqrt{2}} (-1 - 2x) \, dA = \frac{-3}{\sqrt{2}} \frac{\text{Area}(R)}{\sqrt{2}} = \frac{-30}{\sqrt{2}}
\]

Normal to \( R \) comes from eqn for plane: \( x - z = 1 \)

\( \Rightarrow \langle 1, 0, -1 \rangle \) \( \Rightarrow \hat{n} = \frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle \) make unit

\[
\left| \int_C \mathbf{F} \cdot d\mathbf{r} \right| = \frac{30}{\sqrt{2}}
\]
18. The surface $S$ shown below has boundary the circle of radius 2 in the $xz$-plane. With respect to the normal vector field indicated, compute the flux of $\mathbf{G} = (0, 3, 0)$ through $S$. (5 points)

**Backup question:** If you can’t find the exact answer, determine whether the flux is positive, zero, or negative and write that in the answer box for partial credit.

Let $D$ be the disc of radius 2 in the $xz$-plane.

Then $D$ and $S$ together bound a solid region $E$. Then

$$0 = \iiint \operatorname{div} \mathbf{G} = \iint_{\partial E} \mathbf{G} \cdot \mathbf{n} \, dA = \iint_{D} \mathbf{G} \cdot \mathbf{n} \, dA + \iint_{S} \mathbf{G} \cdot \mathbf{n} \, dA.$$

Thus $\iint_{S} \mathbf{G} \cdot \mathbf{n} \, dA = -\iint_{D} \mathbf{G} \cdot \mathbf{n} \, dA$

= $\iint_{D} \mathbf{G} \cdot \mathbf{j} \, dA$

= $\iint_{D} 3 \, dA = 3 \operatorname{Area}(D) = 3 \pi 2^2 = 12 \pi$

Thus $\iint_{S} \mathbf{G} \cdot d\mathbf{s} = 12 \pi$

Scratch Space
19. Suppose \( g(x, y, z) = e^x + y \cos z \) and \( \mathbf{G} = \nabla g = \langle e^x, \cos z, -y \sin z \rangle \) is its gradient vector field. (2 points each)

(a) Let \( \mathbf{C} \) denote the parametric curve \( \mathbf{r}(t) = \langle 2, t, \pi t \rangle \) for \( 0 \leq t \leq 1 \). The integral \( \int_C \mathbf{G} \cdot d\mathbf{r} \) is:

\[
\begin{array}{c}
e^2 & \pi & 1 & 0 & e^2 - 1 \\
\end{array}
\]

(b) Let \( \mathbf{S} \) denote the hemisphere defined by \( x^2 + y^2 + z^2 = 1 \) and \( z \geq 0 \); let \( \mathbf{n} \) denote the upward unit normal. The integral \( \iint_S \text{curl} \mathbf{G} \cdot \mathbf{n} \, dA \) is: \( \text{positive} \) \( \text{zero} \) \( \text{negative} \)

(c) Consider the vector field \( \mathbf{F} = \langle -y, x + z, y \rangle \). The vector field \( \mathbf{F} \) is: \( \text{conservative} \) \( \text{not conservative} \)

20. Suppose \( E \) is a solid region in \( \mathbb{R}^3 \) looking like the picture at right.

(a) Suppose \( E \) has volume 10 and is made of material of constant density. Check the box next to the integral that must compute the \( x \)-coordinate of its center of mass. (2 points)

\[
\begin{array}{c}
\square & \frac{1}{30} \iiint_E \mathbf{A} \cdot dS & \text{for } \mathbf{A} = \langle x, y, z \rangle & \square & \iiint_E x \, dV \\
\checkmark & \frac{1}{20} \iiint_E \mathbf{B} \cdot dS & \text{for } \mathbf{B} = \langle z, xy, xz \rangle & \square & \frac{1}{10} \iiint_E y \, dV \\
\square & \text{None of these.} \\
\end{array}
\]

(b) Assuming the origin lies inside of \( E \), determine the flux of \( \mathbf{H} = \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle \) through \( \partial E \).

(2 points)

\[
\text{Use Gauss's Law: } \quad \iiint_{\partial E} \mathbf{H} \cdot d\mathbf{S} = -12 \pi 
\]