1. Consider the points $A = (2, 0, 1)$ and $B = (4, 2, 5)$ in $\mathbb{R}^3$.

(a) Find the point $M$ which is halfway between $A$ and $B$ on the line segment $L$ joining them. (2 pts)

\[
M = A + \frac{1}{2} \vec{AB} = (2, 0, 1) + \frac{1}{2} (2, 2, 4) = (2, 0, 1) + (1, 1, 2) = (3, 1, 3)
\]

(b) Find the equation for the plane $P$ consisting of all points that are equidistant from $A$ and $B$. (3 pts)

Point on $P$: $M = (3, 1, 3)$

Normal to $P$: $\frac{1}{2} (\overrightarrow{w} - \overrightarrow{v}) = \frac{1}{2} ((4, 2, 5) - (2, 0, 1))$

$= \frac{1}{2} (2, 2, 4) = (1, 1, 2)$

Ans: $1 \cdot (x-3) + 1 \cdot (y-1) + 2 \cdot (z-3) = 0$

Or: $x + y + 2z = 10$

2. Consider the function

\[
f(x, y) = \begin{cases} 
\frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

(a) Compute the following limit, if it exists. (4 pts)

Along $x$-axis: $f(x, 0) = \frac{x \cdot 0}{x^2 + 0^2} = 0$

Along $y = x$: $f(x, x) = \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2}$

So as $f$ does not approach a consistent value as $(x, y) \to (0, 0)$, the limit does not exist.

(b) Where on $\mathbb{R}^2$ is the function $f$ continuous? (1 pts)

Everywhere except $(0, 0)$.

Note: $f$ is not cont. at $(0, 0)$ b/c $\lim_{(x, y) \to (0, 0)} f(x, y)$ doesn't exist.
3. Match the following functions $\mathbb{R}^2 \rightarrow \mathbb{R}$ with their graphs and contour diagrams. Here each contour diagram consists of level sets $\{f(x, y) = c_i\}$ drawn for evenly spaced $c_i$. (9 pts)

(a) $\sqrt{8 - 2x^2 - y^2}$  
(b) $\cos x$  
(c) $xy$

- **a)** $z^2 = 8 - 2x^2 - y^2$
  - $2x^2 + y^2 + z^2 = 8$
  - **Notes:** So, surface is part of an ellipsoid.

- **b)**
  - $y = \text{constant}$
  - $x = \cos (k)$ = const.
  - **Notes:** as $y$ changes, function doesn't change.

- **c)**
  - $x = \text{constant}$
  - $z = kx$

- **d)** $k^2 = 8 - 2x^2 - y^2$
  - $2x^2 + y^2 = 8 - k^2$ -> ellipse

- **e)** $xy = k$  
  - **Notes:** if $z = k$, $\cos x = k$  
  - **Notes:** $x$ is constant.
4. Consider the function \( f: \mathbb{R}^2 \to \mathbb{R} \) given by \( f(x, y) = xy \).

(a) Use Lagrange multipliers to find the global (absolute) max and min of \( f \) on the circle
\( x^2 + y^2 = 2 \). (6 pts)

First \( \nabla f = \lambda \nabla g \)

\[ \begin{align*}
(\lambda, \lambda) &= \nabla f = \nabla g = \lambda (2x, 2y).
\end{align*} \]

Thus \( y = 2\lambda x \) and \( x = 2\lambda y \) and so \( 2\lambda = \frac{x}{y} = \frac{y}{x} \).

Therefore \( \lambda^2 = y^2 \). Since also \( x^2 + y^2 = 2 \), we get \( 2\lambda^2 = 2 \)
\( \Rightarrow \lambda^2 = 1 \) and \( y^2 = 1 \). So there are 4 critical pts.

<table>
<thead>
<tr>
<th>Point</th>
<th>(1, 1)</th>
<th>(1, -1)</th>
<th>(-1, 1)</th>
<th>(-1, -1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Val of ( f )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>Type</td>
<td>Max</td>
<td>Min</td>
<td>Min</td>
<td>Max</td>
</tr>
</tbody>
</table>

Note: Global min/max exist since \( C \) is closed and bounded.

(b) If they exist, find the global min and max of \( f \) on \( D = \{ x^2 + y^2 \leq 2 \} \). (2 pts)

Need to check for critical pts. of \( f \) inside \( D \). These occur when \( \nabla f = (0, 0) \)

\( \nabla f = (2x, 2y) = (0, 0) \)
\( \Rightarrow (x, y) = (0, 0) \).

Note: we don't need to classify this critical pt b/c EVT tells us max & min must both occur on \( D \) either at a pt on boundary, or at a critical pt inside \( D \).

\( f(0,0) = 0 \), since \( 1 > 0 \). -1 < 0,
1 is global max \(-1 \) is global min.

(c) For each critical point in the interior of \( D \) you found in part (b), classify it as a local min, local max, or saddle. (2 pt)

\( \begin{align*}
\frac{\partial^2 f}{\partial x^2} &= y, \\
\frac{\partial^2 f}{\partial y^2} &= x, \\
\frac{\partial^2 f}{\partial x \partial y} &= 0.
\end{align*} \)

\( \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0 \) hence a saddle.

(d) If they exist, find the global min and max of \( f \) on \( \mathbb{R}^2 \). (2 pts)

Neither exist since \( f \to +\infty \) along \( y = x \)
and \( -\infty \) along \( y = -x \). (\( f \) is surface from 3 (c))
5. A function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) takes on the values shown in the table at right.

(a) Estimate the partials \( f_x(1, 1) \) and \( f_y(1, 1) \). (2 pts)

\[
\begin{align*}
f_x(1, 1) & \approx \frac{f(1.4, 1) - f(1, 1)}{0.4} = \frac{3.4 - 3.0}{0.4} = 1 \\
f_y(1, 1) & \approx \frac{f(1, 1.4) - f(1, 1)}{0.4} = \frac{3.8 - 3.0}{0.4} = 2
\end{align*}
\]

(b) Use your answer in (a) to approximate \( f(1.1, 1.2) \). (2 pts)

\[
f(1.1, 1.2) \approx f(1, 1) + f_x(1, 1)(0.1) + f_y(1, 1)(0.2) \\
\approx 3 + 1(0.1) + 2(0.2) = 3.5.
\]

(c) Determine the sign of \( f_{xy}(1, 1) \): 

- positive
- negative
- zero

\[
f_y(1.4, 1) \approx \frac{f(1.4, 1.4) - f(1.4, 1)}{0.4} = \frac{4.36 - 3.40}{0.4} = \frac{0.96}{0.4} > 2.
\]

Thus \( f_y \) increases as \( x \) increases \( \Rightarrow \frac{\partial}{\partial x} f_y > 0 \).

6. Consider the region \( E \) shown at right, which is bounded by the \( xy \)-plane, the plane \( z = y = 0 \) and the surface \( x^2 + y = 1 \). Complete setup, but do not evaluate, a triple integral that computes the volume of \( E \). (6 pts)

\[
\int_{-1}^{1} \int_{0}^{\sqrt{1 - y^2}} \int_{0}^{1} dz \; dy \; dx
\]

or

\[
\int_{0}^{1} \int_{-\sqrt{1 - y^2}}^{\sqrt{1 - y^2}} \int_{0}^{1} dz \; dx \; dy
\]
7. Consider the portion $R$ of the cylinder $x^2 + y^2 \leq 2$ which lies in the positive octant and below the plane $z = 1$. Compute the total mass of $R$ when it is composed of material of density $\rho = e^{x^2+y^2}$.

\[ \text{Mass} = \iiint_R \rho \, dV = \iiint_R e^{x^2+y^2} \, dV \]

\[ = \int_0^{\sqrt{2}} \int_0^{\sqrt{2}} \int_0^{\sqrt{1/2}} e^{r^2} \, r \, d\theta \, dz \, dr \]

using cylindrical coord.

\[ = \int_0^\frac{\sqrt{2}}{2} \pi e^{u} \, du \]

\[ = \frac{\pi}{4} (e^2 - 1). \]

8. For the curve $C$ in $\mathbb{R}^2$ shown and the vector field $F = (\ln(\sin(x)), \cos(\sin(y)) + x)$ evaluate $\int_C F \cdot dr$ using the method of your choice. (5 pts)

Let's use Green's Thm:

\[ \int_C F \cdot dr = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]

because $C$ is clockwise

\[ = \iint_D 1 - 0 \, dA \]

\[ = \text{Area}(D) = -\frac{1}{2} \]
9. Let $R$ be the region shown at right.

(a) Find a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ taking $S = [-1,1] \times [-1,1]$ to $R$. (4 pts)

We want $T(1,0) = (1,1) \quad T(0,1) = (-1,1)$. If

$T(u,v) = (au+bv, cu+dv)$ we can solve $T(1,0) = (a,c) = (1,1)
\quad T(0,1) = (b,d) = (-1,1)$

to find $T(u,v) = (u-v, u+v)$.

Check:

$T(1,1) = (1-1, 1+1) = (0,2)
\quad T(1,-1) = (2, 0)$

(b) Use your change of coordinates to evaluate $\iint_R y^2 \, dA$ via an integral over $S$. (6 pts)

**Emergency backup transformation:** If you can’t do (a), pretend you got the answer $T(u,v) = (uv, u+v)$ and do part (b) anyway.

$$
\iint_R y^2 \, dA = \int_1 \int_1 (u+v)^2 \left| \text{det } J \right| \, du \, dv
$$

$T = (g_u, g_v)
\quad J = \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}
\quad \text{det } J = 2
$

$$
= \int_{-1}^1 \int_{-1}^1 2(u^2 + 2uv + v^2) \, du \, dv
= 2 \int_{-1}^1 \frac{u^3}{3} + uv^2 + v^2 u \bigg|_{u=-1}^1 \, dv
= 2 \int_{-1}^1 \frac{2}{3} + 2v^2 \, dv
= \frac{16}{3}.
$$
10. Consider the surface $S$ which is parameterized by $r(u, v) = (\sqrt{1 + u^2} \cos v, \sqrt{1 + u^2} \sin v, u)$ for $-1 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

(a) Circle the picture of $S$. (2 pts)

(b) Completely setup, but do not evaluate, an integral that computes the surface area of $S$. (6 pts)

\[
\text{Area} = \iint_S 1 \, dA = \iint_D |\hat{r}_u \times \hat{r}_v| \, du \, dv
\]

\[
= \int_0^{2\pi} \int_{-1}^1 \sqrt{1 + 2u^2} \, du \, dv
\]

\[
\hat{r}_u \times \hat{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{u}{\sqrt{1+u^2}} \cos v & \frac{u}{\sqrt{1+u^2}} \sin v & 0 \\ \frac{-u^2}{\sqrt{1+u^2}} \sin v & \frac{-u^2}{\sqrt{1+u^2}} \cos v & 1 \end{vmatrix}
\]

\[
= \begin{pmatrix} \frac{u}{\sqrt{1+u^2}} \cos v \\ \frac{-u^2}{\sqrt{1+u^2}} \sin v \\ 0 \end{pmatrix}
\]

\[
|\hat{r}_u \times \hat{r}_v| = \sqrt{(1+u^2) \cos^2 v + (1+u^2) \sin^2 v + u^2}
\]

11. For the cone $S$ at right, give a parameterization $r$: $D \to S$. Explicitly specify the domain $D$. (5 pts)

Params: $u = y$ \quad $v = \text{angle about y-axis}$

Radius about y-axis is a fn of $y = u$

\[
r = \sqrt{1+y^2} \quad r+y=1
\]

so take $D$

\[
y = \begin{cases} 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 2\pi \end{cases}
\]

\[
r(u,v) = (u(1-u) \cos v, u, (1-u) \sin v)
\]
12. Consider the region $R$ in $\mathbb{R}^3$ above the surface $x^2 + y^2 - z = 4$ and below the $xy$-plane. Also consider the vector field $\mathbf{F} = (0, 0, z)$.

(a) Circle the picture of $R$ below. (2 pts)

(b) Directly calculate the flux of $\mathbf{F}$ through the entire surface $\partial R$, with respect to the outward unit normals. (10 pts)

Now $\partial R = \{T = \infty\}$ and $\{S = \infty\}$. First, $\iint_{\partial R} \mathbf{F} \cdot \mathbf{n} \, dA = \iint_{\partial T} (0, 0, 0) \cdot (0, 0, 1) \, dA = \iint_{\partial T} 0 \, dA = 0$. Second, let's parametrize $S$ by $\mathbf{F}(u, v) = (u, v, u^2 + v^2 - 4)$ on $D = \{u^2 + v^2 \leq 4\}$. Then $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \left| \begin{array}{ccc} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{array} \right| = (2u, -2v, 1)$. As this points the wrong way we use $\mathbf{n} = \mathbf{r}_v \times \mathbf{r}_u$ instead. Now $\text{Flux} = \iint_{S} (\mathbf{F} \cdot \mathbf{n}) \, dA = \iint_{D} \mathbf{F}(\mathbf{F}(u, v)) \cdot (\mathbf{r}_v \times \mathbf{r}_u) \, du \, dv = \iint_{D} (0, 0, u^2 + v^2 - 4) \cdot (2u, 2v, -1) \, du \, dv$

$$= \int_{0}^{2\pi} \int_{0}^{\sqrt{4-u^2-v^2}} (4-r^2) r \, dr \, dr = 2\pi \int_{0}^{\sqrt{4}} 4r - r^3 \, dr$$

$$= 2\pi \left( 2r^2 - \frac{r^4}{4} \right) \bigg|_{r=0}^{r=2} = 2\pi (8 - 4) = 8\pi$$

Hence $\iint_{\partial R} \mathbf{F} \cdot \mathbf{n} \, dA = \iint_{\partial T} \mathbf{F} \cdot \mathbf{n} \, dA + \iint_{S} \mathbf{F} \cdot \mathbf{n} \, dA = 0 + 8\pi = 8\pi$

(c) Use the Divergence Theorem and your answer in (b) to compute the volume of $R$. (3 pts)

$$\iiint_{R} \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_{R} \nabla \cdot \mathbf{F} \, dV = \iiint_{R} 1 \, dV = \text{Volume}.$$

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 1$$

So: $\text{Vol} = 8\pi$. 


13. Let $C$ be the curve shown at right, which is the boundary of the portion of the surface $x + z^2 = 1$ in the positive octant where additionally $y \leq 1$.

(a) Label the four corners of $C$ with their $(x, y, z)$-coordinates. (1 pt)

(b) For $F = (0,xyz,xyz)$, directly compute $\int_C F \cdot dr$. (6 pts)

Break $C$ up into $C_2$.

Notice if any of $x$, $y$ or $z$ is 0, then $F = \mathbf{0}$. Thus for any $C_i$ except $C_1$, we have $\int_{C_i} F \cdot dr = \int_{C_i} \mathbf{0} \cdot dr = 0$.

For $C_1$, we parameterize $-C_1$ via $\vec{r}(t) = (1-t^2, 1, t)$ for $0 \leq t \leq 1$.

So $\int_{C_1} F \cdot dr = -\int_{C_1} F \cdot dr = -\int_0^1 (0, t-t^3, t^3) \cdot (-2t, 0, 1) \, dt$

$= \int_0^1 t^3 - t \, dt = \left[ \frac{1}{4} t^4 - \frac{1}{2} t^2 \right]_0^1 = \frac{1}{4}$

(c) Compute curl $F$. (2 pts)

Hence $\int_C F \cdot dr = \sum_{C_i} \int_{C_i} F \cdot dr = 0 + 0 + 0 - \frac{1}{4} = -\frac{1}{4}$

$\text{curl} \, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & xz \end{vmatrix} = (xz-xy, -yz, yz)$

(d) Use Stokes' Theorem to compute the flux of curl $F$ through the surface $S$ where the normals point out from the origin. (3 pts)

$\int(S(\text{curl} \, F)) \cdot \vec{n} \, dA = -\int_C F \cdot dr = \frac{1}{4}$

Since orient of $C$ doesn't mesh with $\vec{n}$.

(e) Give two distinct reasons why the vector field $F$ is not conservative. (2 pts)

$\text{curl} \, F \neq 0$ and $C$ is a closed curve with $\int_C F \cdot dr \neq 0$. 