Lecture 40: More on Stokes' Thm

\[ \text{Last time: Stokes' Thm: } S \text{ surface in } \mathbb{R}^3 \]
\[ \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ vector field. Then} \]
\[ \int_S \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \hat{n} \, dA \]

\[ \text{Ex: } \vec{F} = (-y, x, yz) \quad \text{curl } \vec{F} = (z, 0, 2) \]

\[ \iint_S (\text{curl } \vec{F}) \cdot \hat{n} \, dA = 2\pi = \int_C \vec{F} \cdot d\vec{r} \text{ for all of these!} \]

[Takes some getting used to, is really just Green's Thm/2 d Divergence Thm in disguise...]

Check the easy one:
\[ \iint_D (\text{curl } \vec{F}) \cdot \hat{n} \, dA \]
\[ = \iint_D (z, 0, 2) \cdot (0, 0, 1) \, dA = \iint_D 2 \, dA = 2 \text{Area(O)} = 2\pi \]
Note: Also works when S has several boundary components, [provided they are oriented correctly]

Understanding Curl: Consider a small paddle wheel at P, of radius r.

\[ \vec{F} = \text{fluid flow} \]

Q: How fast does it spin?

A: \[ \omega = \frac{1}{2\pi r^2} \oint_{C_r} \vec{F} \cdot d\vec{r} \]

units = \text{radians / time}

Plausible: \text{ want tangential component of } \vec{F} \text{ as it hits the paddles.}

\text{Looks like an average (almost).}

Stokes says:

\[ \omega = \frac{1}{2\pi r^2} \iint_{D_r} (\text{curl } \vec{F}) \cdot \hat{n} \, dA \]
\[
\frac{1}{2} \left( \frac{1}{\text{Area}(D_r)} \int_{D_r} (\text{curl} \ \vec{F}) \cdot \vec{n} \ dA \right)
\]

Taking \( r \to 0 \) get: \( \omega = \frac{1}{2} (\text{curl} \ \vec{F}) \cdot \vec{n} \)

So the rate of rotation is fastest the direction of \( \text{curl} \ \vec{F}(\rho) \) and then \( \omega = \frac{1}{2} |\text{curl} \ \vec{F}| \).

Note: A vector field where \( \text{curl} \ \vec{F} = \vec{0} \) everywhere are called \underline{irrotational}.

\(\text{Ex:} \ \vec{F} = \frac{1}{x^2+y^2} (-y, x, 0) \) has \( \text{curl} \ \vec{F} = \vec{0} \) except at \((0,0)\) where it's not defined.

Experimentally, a draining tub is an irrotational flow!
Conservative Vector Fields: \( \vec{F}: \mathbb{R}^n \to \mathbb{R}^n \)

is conservative if \( \vec{F} = \nabla f \) for some \( f: \mathbb{R}^n \to \mathbb{R} \)

Ex: \( \vec{F} = (x, y) \) is conservative since

\[ \vec{F} = \nabla \left( \frac{1}{2}(x^2 + y^2) \right) \]

\( \vec{F} = (-y, x) \) is not conservative since

\[ \frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y} \]

Thm A: If \( \vec{F} \) on a connected set \( D \) in \( \mathbb{R}^n \) is conservative if and only if \( \int_C \vec{F} \cdot d\vec{F} = 0 \) for every closed curve \( C \).

Thm B: If \( D \) is simply connected (no holes) then \( \vec{F} = (P, Q) \) is conservative if and only if \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \).

Missing Link: If \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \) then \( \int_C \vec{F} \cdot d\vec{F} = 0 \) for each closed curve \( C \). Hence \( \vec{F} \) is conservative by Theorem A.
Reason: As $D$ has no holes, the curve $C$ is the boundary of a region $R$ where $\mathbf{F}$ makes sense. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0.$$

Next time: What is Thm B for $\mathbb{R}^3$?

Suppose $\mathbf{F} = \nabla f = (f_x, f_y, f_z)$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \left( \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial z \partial y}, 0, 0 \right)$$

Q: Is this enough?