Lecture 3: Dot product (12.3) and lines and planes in $\mathbb{R}^3$ (12.5)

Last time: \( \vec{v} = (v_1, v_2, v_3), \quad \vec{w} = (w_1, w_2, w_3) \)

\[ \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \]

Key: \( \vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta \)

Projection:

\[ \text{proj}_\vec{v} \vec{w} = \text{component of } \vec{w} \text{ along } \vec{v} \]

\[ = \text{scalar mult of } \vec{v} \text{ closest to } \vec{w}. \]

Note: \( |\text{proj}_\vec{v} \vec{w}| = |\vec{w}| \cos \theta \)

So: \( \text{proj}_\vec{v} \vec{w} = |\vec{w}| \cos \theta \left( \begin{array}{c} \text{unit vector} \\ \text{pointing in same dir as } \vec{v} \end{array} \right) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \vec{v} \)
Work = (force) \times (distance)

\begin{align*}
W &= \| \mathbf{F} \| \| \mathbf{d} \| \\
&= |\text{proj}_\mathbf{d} \mathbf{F} | \| \mathbf{d} \| \\
&= \left| \frac{\mathbf{F} \cdot \mathbf{d}}{\| \mathbf{d} \|^2} \right| \| \mathbf{d} \| \\
&= \frac{\mathbf{F} \cdot \mathbf{d}}{\| \mathbf{d} \|} \\
&= \mathbf{F} \cdot \mathbf{d} \\
\end{align*}

\[ W = \mathbf{F} \cdot \mathbf{d} \]

**Regression:**

Roughly, \( y = c \times x \).
What is \( c \)?

\[ \bar{x} = (x_1, x_2, x_3, \ldots, x_n) \quad \bar{y} = (y_1, y_2, y_3, \ldots, y_n) \]
If \( y_i = cx_i \) for each \( i \), then
\[
\hat{y} = c \hat{x}
\]

So "best fit" is
\[
c = \frac{\hat{x} \cdot \hat{y}}{|\hat{x}|^2}
\]

which minimizes \( |\hat{y} - c \hat{x}| \) in general

(model has more parameters, and proj is onto a plane or similar. Cf. Math 415.)

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Planes in \( \mathbb{R}^3 \):

Take a point \( P_0 = (x_0, y_0, z_0) \) on the plane, with \( \vec{r}_0 \) its pos. vector.

Test if \( P = (x, y, z) \) is in the plane:

\( \vec{n} \) and \( \vec{r} - \vec{r}_0 \) are perpendicular (orthogonal)

i.e. \( \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \)
If \( \mathbf{n} = (a, b, c) \), have \( P = (x, y, z) \) in the plane when

\[
0 = (a, b, c) \cdot (x - x_0, y - y_0, z - z_0)
\]

\[
= a(x - x_0) + b(y - y_0) + c(z - z_0)
\]

\[
= ax + by + cz + d
\]

with \( d = -(ax_0 + by_0 + cz_0) \)

Conversely,

\[
ax + by + cz + d = 0
\]

defines a plane [unless \( a = b = c = 0 \)]

[Normal vectors will be a key concept later in the course. Can be used to solve geometric problems about planes]
Example: $P_1$ is the plane given by $x + y + z - 1 = 0$.

Plane is set by 3 points:

Normal vector

$\vec{n}_1 = (1, 1, 1)$

$P_2 = \{ z = 1 \}$

$\vec{n}_2 = (0, 0, 1)$

Q1: What is the intersection of $P_1$ and $P_2$?

Q2: What is the angle between $P_1$ and $P_2$?
A line $L$:

Points on $L$ sat

\[ x + y + z = 1 \quad \text{and} \quad z = 1 \]

Two easy solutions:

\[ \vec{r}_0 = (0, 0, 1) \]
\[ \vec{r} = (1, -1, 1) \]
\[ \vec{v} = \vec{r} - \vec{r}_0 \]
\[ = (1, -1, 0) \]

Every pt on $L$ has the form

\[ \vec{r}(t) = \vec{r}_0 + t \vec{v} \]
and so $L$ is parameterized by

$$(0, 0, 1) + t(1, -1, 0) = (t, -t, 1)$$

This is just like worksheet 3(e) from yesterday
A2: Angle between planes is same as angle between normals.

\[
\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{1}{\sqrt{3} \cdot 1} = \frac{1}{\sqrt{3}}
\]

\[\Rightarrow \theta \approx 54.7^\circ\]