Lecture 17: Introduction to space curves (13.1 and 13.2)

Next two weeks: "Curves in space..."

Chapter 13, Section 16.1 - 16.3

Last time: Not really relevant:

Curve: \( \vec{r}: \mathbb{R} \to \mathbb{R}^2 \) or \( \mathbb{R}^3 \) (vector-valued function)

Example: \( \vec{r}: \mathbb{R} \to \mathbb{R}^2 \)
\[
\vec{r}(t) = (\cos t, \sin t)
\]

Example: \( \vec{r}: \mathbb{R} \to \mathbb{R}^3 \)
\[
\vec{r}(t) = (1, 0, 1) + t(2, 1, -1) = (1 + 2t, t, 1 - t)
\]

Example: \( \vec{r}: \mathbb{R} \to \mathbb{R}^3 \)
\[
\vec{r}(t) = (\cos t, \sin t, t)
\]

Cycloid:

Suppose a wheel with radius 1 rotates at 1 radian/see
Thus after time $t$, the center has moved a distance $t$. So if center is at $(0,1)$ at $t=0$, its at $(t,1)$. Combined with the rotation, we get

$$\vec{r}(t) = (t,1) + (-\sin t, -\cos t)$$

$$= (t - \sin t, -\cos t)$$

The vector is $(-\sin t, -\cos t)$.

Upside down, this curve has the following remarkable properties:

- **Tautochrone**
- **Brachistochrone**

Same time to bottom

Huygens (1650s) pendulunm clock

Swings along a cycloid

This is an infinite dimensional min/max problem.

(“Calculus of Variations”)

Click!
Derivative: \( \vec{r} : \mathbb{R} \to \mathbb{R}^3 \quad \vec{r}(t) = (r_1(t), r_2(t), r_3(t)) \)

\[
\vec{r}'(t) = (r'_1(t), r'_2(t), r'_3(t)) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}
\]

The vector \( \vec{r}'(t) \) is tangent to the curve at \( \vec{r}(t) \). Geometrically, it is the \underline{velocity vector} of the motion at time \( t \).

(Notice the units of \( r'(t) \) are \( \frac{\text{distance}}{\text{time}} \)). The \underline{speed} of the motion at time \( t \) is \( |\vec{r}'(t)| \).

Also \( \vec{r}' : \mathbb{R} \to \mathbb{R}^3 \) and \( \vec{r}''(t) \) is the \underline{acceleration} at time \( t \).
Length of a curve: \( \mathbf{r} : [a, b] \rightarrow \mathbb{R}^2 \) (or \( \mathbb{R}^3 \))

How long is this path?

If we view this as the motion of something, have:

\[
\text{distance traveled} = \int_a^b \text{speed} \ dt = \int_a^b |\mathbf{r}'(t)| \ dt
\]

Another point of view: Approximate by straight segments

\[
\text{Length} \approx \sum_{i=0}^{n} \text{length of segments} = \sum_{i=0}^{n} |\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)|
\]
Linear approximation: (just like in one var.)

\[ \mathbf{r}(t_i + \Delta t) \approx \mathbf{r}(t_i) + \mathbf{r}'(t_i) \Delta t \]

Take \( \Delta t \)

and so

\[ |\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)| \approx |\mathbf{r}'(t_i)| \Delta t. \]

So

\[ \text{Length} \sim \sum_{i=0}^{n} |\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)| \approx \sum_{i=0}^{n} |\mathbf{r}'(t_i)| \Delta t \]

Riemann Sum

As \( \Delta t \to 0 \), these Riemann sums converge to

\[ \int_{a}^{b} |\mathbf{r}'(t)| \, dt, \]

and also to the length.
Ex: Cycloid

\[ \vec{r}(t) = (t - \sin t, 1 - \cos t) \]

\[ \vec{r}'(t) = (1 - \cos t, \sin t) \]

\[ |\vec{r}'(t)| = \sqrt{(1 - \cos t)^2 + \sin^2 t} \]

\[ = \sqrt{2 - 2 \cos t} \]

Length = \[ \int_0^{2\pi} |\vec{r}'(t)| \, dt = \int_0^{2\pi} \sqrt{2 - 2 \cos t} \, dt \]

\[ = \int_0^{2\pi} \sqrt{2 \sin^2 \frac{t}{2}} \, dt \]

\[ = \int_0^{2\pi} 2 \sin \frac{t}{2} \, dt \quad \text{(since } \sin \frac{t}{2} \text{ is } > 0 \text{ on } [0, 2\pi]) \]

\[ = -4 \cos \frac{t}{2} \bigg|_0^{2\pi} = 4 - (-4) = 8. \]