Tuesday, November 29  ** Surface integrals of vector fields and related theorems

1. Consider the region $D$ in $\mathbb{R}^3$ bounded by the $x$-$y$-plane and the surface $x^2 + y^2 + z = 1$.

   (a) Make a sketch of $D$.

   (b) The boundary of $D$, denoted $\partial D$, has two parts: the curved top $S_1$ and the flat bottom $S_2$. Parameterize $S_1$ and calculate the flux of $F = (0, 0, z)$ through $S_1$ with respect to the upward pointing unit normal vector field. Check your answer with the instructor.

   (c) Without doing the full calculation, determine the flux of $F$ through $S_2$ with the downward pointing normals.

   (d) Determine the flux of $F$ through $\partial D$ with the outward pointing normals.

   (e) Apply the Divergence Theorem and your answer in (d) to find the volume of $D$. Check your answer with the instructor.

2. Consider the vector field $F = (-y, x, z)$.

   (a) Compute curl $F$.

   (b) For the surface $S_1$ above, evaluate $\int_{S_1} (\text{curl } F) \cdot \mathbf{n} \ dA$.

   (c) Check your answer in (b) using Stokes' Theorem.

3. If time remains:

   (a) Check your answer in 1(e) by directly calculating the volume of $D$.

   (b) Repeat 2 (b-c) for the surface $S_2$ and also for the surface $\partial D$. What exactly does 2(c) mean for the surface $\partial D$?

   (c) For the vector field $F = (-y, x, z)$ from the second problem, compute div(curl $F$). Now suppose $F = (F_1, F_2, F_3)$ is an arbitrary vector field. Can you say anything about the function div(curl $F$)?