Last time: \( E: \mathbb{R} \to \mathbb{R} \)

Say \( \lim_{h \to 0} E(h) = 0 \) if given \( \varepsilon > 0 \) can always find a \( \delta > 0 \) so that when 
\[ 0 < |h| < \delta \] then \( |E(h)| < \varepsilon \)

[Foreshadow story of the Sleipner A sinking.]

Now suppose \( E: \mathbb{R}^2 \to \mathbb{R} \). We say \( \lim_{\vec{h} \to 0} E(\vec{h}) = 0 \)
if given \( \varepsilon > 0 \) can always find \( \delta > 0 \) so that 
when \( 0 < |\vec{h}| < \delta \) then \( |E(\vec{h})| < \varepsilon \).

**Ex:**

\[ \lim_{(x,y) \to (0,0)} x + 3y = 0 \]

**Reason:** Let \( \varepsilon > 0 \) be given. Take \( \delta = \varepsilon/4 \).

If \( h = (x,y) \) and \( |h| < \delta \), then \( |x| < \delta \) and \( |y| < \delta \) as shown.
Then
\[ |x + 3y| < |x| + |13y| < 8 + 3\delta = 4\delta = \varepsilon. \]

More generally, if \( f: \mathbb{R}^2 \to \mathbb{R} \) then
\[ \lim_{x \to \bar{a}} f(x) = c \quad \text{if} \quad f(\bar{a} + h) = c + E(h) \]

where \( \lim_{\dot{h} \to \bar{h}} E(h) = 0. \)

Same notion of limit works for \( f: \mathbb{R}^3 \to \mathbb{R} \)
or \( \mathbb{R}^n \to \mathbb{R} \) or even \( \mathbb{R}^n \to \mathbb{R}^m. \)

A more complicated example:

Take \( f(x, y) = \frac{2xy}{x^2 + y^2} \) not defined at \((0,0)\).

What is
\[ \lim_{(x,y) \to \bar{a}} f(x, y) = ? \]

First, try our usual trick: reduce the dimension
Along the x-axis:

\[ f(x,0) = \frac{2x \cdot 0}{x^2 + 0^2} = 0 \]

which suggests \( \lim = 0 \). But along the line \( y = x \)

\[ f(x,x) = \frac{2x \cdot x}{x^2 + x^2} = 1 \quad \text{(for } x \neq 0) \]

Thus the limit does not exist.

Can actually work out the graph of \( f \).

Along the line \( y = cx \) have:

\[ f(x,cx) = \frac{2x \cdot (cx)}{x^2 + (cx)^2} = \frac{2cx^2}{x^2 + c^2 x^2} = \frac{2c}{1 + c^2} \]

E.g. \( f = -1 \) along \( y = -x \)

The full picture is:
Even odder:

\[ f(x,y) = \frac{xy^2}{x^2 + y^4} \]

What is limit as \((x,y) \to 0\)?

Along the line \(y = cx\) we have

\[ f(x,cx) = \frac{x(cx)^2}{x^2 + (cx)^4} = \frac{cx^3}{x^2(1+c^4x^2)} = \frac{cx}{1+c^4x^2} \]

which \(\to 0\) as \(x \to 0\).
But: Along $x = y^2$ we have

$$f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}.$$ 

So again, the limit does not exist!

**Rules for limits:**

$f, g : \mathbb{R}^2 \to \mathbb{R}$

(a) \[ \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \]

(b) \[ \lim_{x \to a} f(x)g(x) = \left( \lim_{x \to a} f(x) \right) \left( \lim_{x \to a} g(x) \right) \]

In both cases, this is provided the RHS all makes sense.