Lecture 16: Constrained min/max (§14.8)

Last time:

** Extreme Value Thm:** $f$ continuous on $D$ in $\mathbb{R}^n$.

If $D$ is closed and bounded, then $f$ has both an absolute min and an absolute max on $D$. These occur at critical pts of $f$ or on the boundary of $D$.

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**Ex:** Find the max of $f(x,y) = x^2 - y^2$ on the unit circle $x^2 + y^2 = 1$.

Start with:

Focus here

\[ f = 0 \]

\[ f = \frac{1}{2} \]

\[ f = \frac{2}{3} \]

\[ x^2 + y^2 = 1 \]

\[ f = \frac{1}{4}, \ f = \frac{1}{2} \]

\[ f = -\frac{2}{3} \]
When we have this picture, don't have a local max since can increase $f$ by moving clockwise along the circle.

However, when the level set of $f$ is tangent to the circle, we can have a local max.

Here, $f$ decreases as we move away from $(1,0)$ in either direction.

Four tangencies in this example:

<table>
<thead>
<tr>
<th>Value of $f$</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>max</td>
</tr>
<tr>
<td>1</td>
<td>max</td>
</tr>
<tr>
<td>-1</td>
<td>min</td>
</tr>
<tr>
<td>-1</td>
<td>min</td>
</tr>
</tbody>
</table>

$(1,0)$  $(0,1)$  $(0,-1)$

absolute in each case, the circle is closed and bounded.
Finding these tangencies in general:

View the circle as the level set \( g(x,y) = 1 \) where \( g(x,y) = x^2 + y^2 \). At a tangency, \( \nabla f \) is at right angles to the circle:

\[ \nabla f \text{ and } \nabla g \text{ point in the same direction:} \]
\[ \nabla f = \lambda \nabla g \]

Key: At a tangency, \( \nabla f \) and \( \nabla g \) point in the same direction:
\[ \nabla f = \lambda \nabla g \]

\[ \nabla f = (2x, -2y) = \lambda (2x, 2y) = (2\lambda x, 2\lambda y) \]

So \( 2x = 2\lambda x \) and \( -2y = 2\lambda y \).

If \( x \neq 0 \), then \( \lambda = 1 \) and \( y = 0 \Rightarrow x = \pm 1 \).
If \( y \neq 0 \), then \( \lambda = -1 \) and \( x = 0 \Rightarrow y = \pm 1 \)

from \( x^2 + y^2 = 1 \).
So the critical points are \((1, 0)\), \((-1, 0)\), \((0, 1)\), \((0, -1)\), just as we found before.

Ex: Find the distance from \(x - y + 2z = 3\) to \(\overrightarrow{0}\)

Minimize: \(f(x, y, z) = x^2 + y^2 + z^2\)
Subject to: \(g(x, y, z) = 3\)

Crit pts: \(\nabla f = \lambda \nabla g = (\lambda, -\lambda, 2\lambda)\)

\(\nabla f = (2x, 2y, 2z)\)

\(\nabla g = (1, -1, 2)\)

\(2x = \lambda\)
\(2y = -\lambda\)
\(2z = 2\lambda\)

Combine with \(g = 3\) gives \(x - (-x) + 2(2x) = 3\)
and so \(x = \frac{1}{2}\), and \(y = -\frac{1}{2}\), \(z = 1\) just as on Monday.

[Key Features:]
1. Algebra easier than before.
2. Don't have to solve for one var, which we won't be able to do for complicated \(g\).]
Ex: Find the rectangular box of area 6 and largest volume. [What do you expect the ans to be?] 

Maximize: \( V = lwh \)

Subject to: \( A = 2lh + 2lw + 2wh = 6 \).

\[ \nabla V = (wh, lh, lw) = \lambda \nabla A \]

\[ = \lambda 2(h+w, l+h, l+w) \]

\[ \Rightarrow \frac{1}{2\lambda} = \frac{1}{w} + \frac{1}{h} = \frac{1}{h} + \frac{1}{l} = \frac{1}{w} + \frac{1}{l} \]

\[ \Rightarrow \frac{1}{l} = \frac{1}{w} = \frac{1}{h} \Rightarrow l = w = h \]

Combine with \( A = 6 \) gives \( 6l^2 = 6 \Rightarrow l = w = h = 1 \).

**Q:** Why does this crit pt have to be a max? 

**Point:** \( V \) and \( A \) is very symmetric. In particular, if \((l_0, w_0, h_0)\) is a crit pt, so is \((w_0, h_0, l_0)\) and \((w_0, l_0, h_0)\). Hence if there is only one crit pt, it must be symmetric.