Lecture 14: intro to min/max (14.7)

One Var:

- Min/max occur at critical pts where $f'(x) = 0$
- Local vs. Absolute extrema
- 2nd derivative test:
  \[
  \begin{cases}
  f''(a) < 0 \Rightarrow \text{max} \\
  f''(a) > 0 \Rightarrow \text{min}
  \end{cases}
  \]

[Need to do the same for $f : \mathbb{R}^n \to \mathbb{R}$, start with]

For $f : \mathbb{R}^2 \to \mathbb{R}$, $(a,b)$ is a critical pt when

\[
\nabla f(a,b) = \overrightarrow{0}
\]

Every local extrema occurs at a critical pt.

Today: 2nd derivative test for fun of two vars.

Tale of 3 critical pts at $(0,0)$

\[
\begin{align*}
  f(x,y) &= x^2 + y^2 \\
  \nabla f &= (2x, 2y) \\
  &\text{Local Min}
  \\
  f(x,y) &= -x^2 - y^2 \\
  \nabla f &= (-2x, -2y) \\
  &\text{Local Max}
  \\
  f(x,y) &= x^2 - y^2 \\
  \nabla f &= (2x, -2y) \\
  &\text{Saddle}
\end{align*}
\]
In one var, also a neither case, e.g. \( f(x) = x^3 \)
\[
\begin{align*}
  f'(x) &= 3x^2 \\
  f''(x) &= 6x
\end{align*}
\]
both 0 at 0.

This was pretty rare, because it's not "stable".

E.g. \( f(x) = x^3 + \frac{1}{10} x \) and \( x^3 - \frac{1}{10} x \) don't have this issue.

However, saddles are stable.

One pt of view on the 2\textsuperscript{nd} der. test in one var.

**Taylor series:** Near \( x_0 \), usually have

\[
f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2} h^2 + E(h)
\]

where \( E(h) \) is really small, i.e. \( \lim_{h \to 0} \frac{E(h)}{h^2} = 0 \).
If \( x_0 \) is a critical pt, then
\[
f(x_0 + h) = f(x_0) + \frac{f''(x_0)}{2} h^2 + E(h)
\]
So near \( x_0 \) the graph of \( f \) looks like
\[
f''(x_0) > 0 \quad \text{local min}
\]
\[
f''(x_0) < 0 \quad \text{local max}
\]

**Taylor series for \( f(x,y) \):** For nice functions, have
\[
f(x_0 + h, y_0 + k) = f(x_0, y_0) + f_x(x_0, y_0) h + f_y(x_0, y_0) k + \frac{f''_{xx}(x_0, y_0)}{2} h^2 + \frac{f''_{yy}(x_0, y_0)}{2} k^2 + E(h, k)
\]

next level of accuracy.

smaller than other terms.
Q: What are $a, b, c$?

A: 
\[ a = \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (x_0, y_0), \quad b = \frac{\partial^2 f}{\partial x \partial y} (x_0, y_0), \quad c = \frac{1}{2} \frac{\partial^2 f}{\partial y^2} (x_0, y_0) \]

Reason: Take $x_0 = y_0 = 0$, $x = h$, $y = k$

\[ f(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + ax^2 + bxy + cy^2 \]

1. \[ f(0, 0) = g(0, 0) \]

2. \[ f_x(0, 0) = g_x(0, 0) \] since \[ g_x(x, y) = f_x(0, 0) + 2ax + by \]

3. Wanted \[ f_{xx}(0, 0) = g_{xx}(0, 0) \]

as well. Since \[ g_{xx} = 2a \] this gives \[ a = \frac{1}{2} f_{xx}(0, 0) \].
Ex: Why Taylor series are so useful:

\[ f(x, y) = \sin \left( \sqrt{1 + \frac{x^2}{2 + \cos y}} - e^{-xy} \right) \]

Now (0,0) is a critical point of this mess, but is it a max? Taylor series is:

\[ f(x, y) = \frac{1}{6} x^2 + xy + E(x, y) \]

Look along lines:

\[ f(x, x) = \frac{7}{6} x^2 + E(x, x) \]

\[ f(x, -x) = \frac{1}{6} x^2 + x(-x) + E(x, -x) = -\frac{5}{6} x^2 + E(x, -x) \]

So \( f \) has a saddle at (0,0).

2nd derivative test: Suppose (a, b) is a critical point of \( f \).

Set

\[ D = \begin{vmatrix}
    f_{xx}(a, b) & f_{xy}(a, b) \\
    f_{xy}(a, b) & f_{yy}(a, b)
\end{vmatrix} \]
If \( D > 0 \) and \( f_{xx}(a,b) > 0 \) then \((a,b)\) is a local min.

If \( D > 0 \) and \( f_{xx}(a,b) < 0 \) then \((a,b)\) is a local max.

If \( D < 0 \) then \((a,b)\) is a saddle.

If \( D = 0 \) or \( f_{xx}(a,b) = 0 \), break glass...

\[ f(x,y) = x^2 + y^2 \quad f_x = 2x \quad f_y = 2y \]

At \((0,0)\) have
\[
D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \quad \text{and} \quad f_{xx}(0,0) = 2 > 0
\]

so a \( \boxed{\text{min}} \)

\[ f(x,y) = -x^2 - y^2 \quad D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0 \quad \text{and} \quad f_{xx}(0,0) < 0
\]

so a \( \boxed{\text{max}} \)

\[ f(x,y) = x^2 - y^2 \quad D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0 \]

\{ \text{Saddles at } (0,0) \}

\[ f(x,y) = xy \quad D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -4 < 0 \]

From worksheet.

Under the hood: Changing Coordinates, learn more about on Tuesday.

Higher diving: Eigenvalues etc...