Lecture 19: Vector fields (16.1 and 16.2)

Last time:
\[ C \subset \mathbb{R}^3 \]
\[ f: \mathbb{R}^3 \to \mathbb{R} \]
\[ \int_C f \, ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| \, dt \]
where \( \vec{r}: [a,b] \to \mathbb{R}^3 \) is a param. of \( C \).

Meanings:
1. Area of \( f \) on \( C \) is \( \frac{1}{\text{len}(C)} \int_C f \, ds \)
2. Total mass = \( \int_C f \, ds \)
3. Area of \( f \) where \( f \) is a density fn

\[ \int_C f \, ds \]

Note: \( ds = |\vec{r}'(t)| \, dt \) is called the "arc length" element.

Also \( \int_C 1 \, ds = \text{Length}(C) \).

Vector Fields: (16.1 and 16.2)

For \( \mathbb{R}^2 \), a vector field is a function \( \vec{F}: \mathbb{R}^2 \to \mathbb{R}^2 \)

Ex: \( \vec{F}(x,y) = -y \hat{i} + x \hat{j} \)

Uses:
- Wind speed/direction
- Fluid flow
- Force magnitude/direction
- Electric/magnetic fields
**Ex: Gravity:**

Large mass \( M \) at \((0,0)\).

Force \( \vec{F} \) on small mass \( m \) depends on position.

\( \vec{F} \) points in direction \(-\hat{r} \)

**Newton's Law:**

\[
|\vec{F}| = \frac{MmG}{|\vec{r}|^2}
\]

If \( \vec{F} = -c \vec{r} \) then \( |\vec{F}| = c |\vec{r}| \implies c = \frac{MmG}{|\vec{r}|^3} \)

So

\[
\vec{F} = -\frac{MmG}{|\vec{r}|^3} \hat{r}
\]

For several bodies, add vector fields.

\[
\vec{F}(x,y) = \vec{F}_1(x,y) + \vec{F}_2(x,y)
\]
Ex: Electric field. Force on a charge $q$ at $(x,y)$ is

$$\vec{F} = \frac{E(x,y)}{q}$$

Where have we seen vector fields before?

A. The gradient!

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$ then $\nabla f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Ex. $f(x,y) = x^2 + y^2$

$\nabla f = (2x, 2y)$

Ex: $f(x,y) = \frac{MmG}{\sqrt{x^2 + y^2}} = \frac{MmG}{1 \cdot r}$

$\nabla f = MmG \left( -\frac{1}{2} (x^2 + y^2)^{-3/2}, 2x \right)$

$= MmG \left( \frac{x}{(x^2 + y^2)^{3/2}}, \frac{y}{(x^2 + y^2)^{3/2}} \right) = \frac{MmG}{1 \cdot r^3} \vec{r} = \vec{F}$

Say that $f$ is a potential function for $\vec{F}$, and that $\vec{F}$ is a conservative vector field.

Think potential energy.
Q: Is $\vec{F} = y\vec{i} - x\vec{j}$ conservative?

A: No. Suppose $\nabla f = \vec{F}$. Since $\vec{F}$ is tangent to the unit circle, following the circle increases $f$. But going all the way around, we end up atback at $(1,0)$.

Integrating Vector Fields: (16.2)

Recall: Work done by gravity

$W = \vec{F} \cdot \vec{d}$

[Assumes constant force.]

How much work does gravity do here?

Motion of ship $\vec{F}: \mathbb{R} \to \mathbb{R}^2$.

Force of grav = vector field $\vec{F}$. 

Break into segments

\[ \mathbf{F}(\mathbf{r}(t)) \]

\[ \mathbf{F}(\mathbf{r}(t_i)) \]

\[ \approx \Delta t \mathbf{F}'(t_i) \]

Work done here \( \approx \mathbf{F}(\mathbf{r}(t_i)) \cdot (\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)) \)

\[ \approx (\mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i)) \Delta t \]

Sum up and take \( \Delta t \to 0 \) to get

\[ \text{Total Work} = \int_a^b \mathbf{F}(\mathbf{F}(t)) \cdot \mathbf{r}'(t) \, dt \]

**General Setup:** \( C \) a curve in \( \mathbb{R}^n \)

\( \mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n \) a vector field.

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{F}(t)) \cdot \mathbf{r}'(t) \, dt \]

for any param \( \mathbf{r}: [a,b] \to \mathbb{R}^n \).

[Note: Answer only depends on direction of param.]
Ex:  \( C = \)\( \text{parabola } y = x^2 \)\( \Rightarrow (1, 1) \) \( \vec{r}(t) = (t, t^2) \) for \( 0 \leq t \leq 1 \).
\( \vec{r}'(t) = (1, 2t) \)

\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^1 (t^2, 1) \cdot (1, 2t) \, dt \]

\[ = \int_0^1 t^2 + 2t \, dt = \left[ \frac{t^3}{3} + t^2 \right]_0^1 = \frac{4}{3}. \]

Explain why this is consistent with the work integral.