Lecture 18: Integration along curves (13.3 and 16.2)

Last time: \( \vec{r} : \mathbb{R} \to \mathbb{R}^2 \text{ or } \mathbb{R}^3 \)

Length: \( \vec{r} : [a, b] \to \mathbb{R}^3 \)

\[ \text{Length} = \int_a^b |\vec{r}'(t)| \, dt \]

Really two concepts:

- Curve: A set of pts in \( \mathbb{R}^3 \) looking like

- Parameterization: \( \vec{r} : \mathbb{R} \to \mathbb{R}^3 \)

  \[ \begin{align*}
  & t = 1 \\
  & t = 0 \\
  & t = -1
  \end{align*} \]

[Any curve has many parameterizations.]

Ex:

\( C = \frac{1}{4} \) of a unit circle in \( \mathbb{R}^2 \)

\( \vec{r} : [0, \frac{\pi}{2}] \to \mathbb{R}^2 \)

\( \vec{r}(t) = (\cos t, \sin t) \)

\( \vec{r}'(t) = (-\sin t, \cos t) \)

\( |\vec{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1 \)

\[ \text{Length} = \int_0^{\frac{\pi}{2}} |\vec{r}'(t)| \, dt = \int_0^{\frac{\pi}{2}} 1 \, dt = \frac{\pi}{2} \]
\( \mathbf{r} : [0,1] \to \mathbb{R} \)

\[ \mathbf{r}(t) = (t, \sqrt{1-t^2}) \]

\[ \mathbf{r}'(t) = (1, \frac{-t}{\sqrt{1-t^2}}) \]

\[ \mathbf{r}''(t) = \sqrt{1 + \frac{t^2}{1-t^2}} = \frac{1}{\sqrt{1-t^2}} \]

Length

\[ = \int_0^1 \frac{1}{\sqrt{1-t^2}} = \arcsin(t) \bigg|_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2} \checkmark \]

Integration along a curve: (Sect 16.2)

\[ C \text{ curve in } \mathbb{R}^2 \]

\[ f : \mathbb{R}^2 \to \mathbb{R} \text{ a fn} \]

\[ \int_C f\, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt \]

where \( \mathbf{r} : [a,b] \to \mathbb{R}^2 \) is a param. of \( C \).

[turns out not to depend on \( \mathbf{r} \), just \( C \).]

Some meanings:

1. \( f = \text{temperature} \)

\[ \text{Average temp along } C = \frac{1}{\text{Length } C} \int_C f\, ds \]

Compare: Average of \( f : \mathbb{R} \to \mathbb{R} \) on \( [a,b] \)

\[ \text{is } \frac{1}{b-a} \int_a^b f(x) \, dx \]
(2) \( f = \text{density of material} \) (mass/length)

\[
\text{Mass of curve} = \int_C f \, ds
\]

(3) Area of region above \( C \) and below the graph of \( f \)

Example: Find arclength of \( f(x,y) = x \) on
\[
\vec{r}(t) = (\cos t, \sin t) \quad 0 \leq t \leq \frac{\pi}{2}
\]

\[
\int_C f \, ds = \int_0^{\pi/2} f(\vec{r}(t)) \left| \vec{r}'(t) \right| \, dt
\]

\[
= \int_0^{\pi/2} \cos(t) \cdot 1 - \sin(t) \bigg|_{t=0}^{\pi/2} = 1 - 0 = 1
\]
Average = \int_C f ds \over \text{Length} = \frac{2}{\pi} \approx 0.6366

\quad \text{Linear Approximation: } \vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3

\vec{r}(t+\Delta t) = \vec{r}(t) + \Delta t \vec{r}'(t) + E(t)

Provided \vec{r}'(t) \text{ exists, will have }

\lim_{t \rightarrow 0} \frac{E(t)}{t} = 0

\text{Alternatively, } \vec{r}(t+\Delta t) - \vec{r}(t) \approx \Delta t \vec{r}'(t)

Understanding these integrals.

\text{Ex.}

\begin{align*}
\text{Average of } f \text{ on } C &= \left( \text{portion when } f=2 \right) \cdot 2 + \left( \text{portion when } f=3 \right) \cdot 3 \\
&= \left( \frac{3}{8} \right) \cdot 2 + \left( \frac{5}{8} \right) \cdot 3 \\
&= \frac{21}{8} = 2 \frac{5}{8}
\end{align*}
More complicated fn:

If $\Delta t$ is small and $f$ is continuous, then $f$ is almost constant on $t_i$ to $t_{i+1}$.

So this segment contributes

$$\approx \frac{\text{length of segment}}{\text{length of } C} \cdot f(\vec{r}(t_i))$$

... to the average. Now the seg. has length $\approx |\vec{r}'(t_i)|\Delta t$ and so

$$\text{Average} \approx \sum_{i=0}^{n-1} \frac{\text{len of } i^{th} \text{ seg.}}{\text{len of } C} f(\vec{r}(t_i))$$

$$\approx \frac{1}{\text{len}(C)} \sum_{i=0}^{n-1} f(\vec{r}(t_i)) |\vec{r}'(t_i)|\Delta t$$
As $\Delta t \to 0$, we get

$$\text{Average} = \frac{1}{\text{len}(C)} \int_0^b f(r(t)) \left| F'(t) \right| \, dt$$

$$= \frac{1}{\text{len}(C)} \int_C f \, ds.$$ 

Compare: $\text{Average of } f \text{ on } [a,b] = \frac{1}{b-a} \int_a^b f(t) \, dt$. 

![average](image)