Lecture 43: Maxwell's Equations

Last time: Charges $Q_i$ at positions $\vec{P}_i$.

$$\vec{E}(\vec{r}) = \sum Q_i \frac{1}{4\pi\varepsilon_0 |\vec{r}-\vec{P}_i|^3} (\vec{r}-\vec{P}_i)$$

Gauss's Law: $D$ a region in $\mathbb{R}^3$

$$\oiint_{\partial D} (\vec{E} \cdot \hat{n}) \, dA = \frac{1}{\varepsilon_0} \text{(Total charge in } D\text{)}$$

On HW for today, used this compute the total charge from a formula for $\vec{E}$. This is not in practical...

When there are many (e.g. $10^{20}$) can't use the formula for $\vec{E}$ directly. Don't want to focus on each charge individually just as we don't count molecules when measuring the mass.

Mass: $\rho(x, y, z)$ mass density, units $= \frac{g}{m^3}$

$$\text{Total Mass} = \iiint_D \rho \, dV$$

Charge: $\rho(x, y, z)$ change density, units $= \frac{\text{coulomb}}{m^3}$

$$\text{Total Charge} = \iiint_D \rho \, dV$$
Q: How does \( \rho \) determine \( \vec{E} \)?

Gauss's Law should still hold, so for a region \( R \) we have

\[
\frac{1}{\varepsilon_0} \left( \text{Charge in } R \right) = \iiint_{\partial R} (\vec{E} \cdot \hat{n}) \, dA = \iiint_{R} \text{Div } \vec{E} \, dV
\]

Divergence Thm

\[
\frac{1}{\varepsilon_0} \iiint_{R} \rho \, dV
\]

As true for all regions \( R \), must have \( \text{div } \vec{E} = \frac{\rho}{\varepsilon_0} \).

Q: Does this answer the question? Not completely since many vector fields have the same divergence.

A. \( \vec{E}(\vec{r}) = (E_1(\vec{r}), E_2(\vec{r}), E_3(\vec{r})) \) where if \( \vec{r} = (a, b, c) \)

then

\[
E_1(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \iiint_{D} \frac{(a-x) \rho(x,y,z)}{|\vec{r}-(x,y,z)|^3} \, dV
\]

etc.

Exercise: Take \( D = \text{unit sphere} \) and \( \rho = 1 \).

Use above to calculate the electric field at \( \vec{r} = (a,b,c) \)

Hint: Use symmetry to reduce to the case where \( \vec{r} = (a,0,0) \).
Maxwell's Equations:

\[ \vec{E}(x, y, z, t) \] - Electric field (at time \( t \)) \((\mathbb{R}^4 \to \mathbb{R}^3)\)

\[ \vec{B}(x, y, z, t) \] - Magnetic field

\( \rho(x, y, z, t) \) - charge density \((\mathbb{R}^4 \to \mathbb{R})\)

Gauss's Law:

\[
\text{div} \vec{E} = \frac{\rho}{\varepsilon_0} \\
\iiint_{\mathcal{R}} \vec{E} \cdot \hat{n} \, dA = \iiint_{\mathcal{R}} \rho \, dV
\]

Gauss's Law for magnetic fields: \([\text{No magnetic monopoles.}]\)

\[
\text{div} \vec{B} = 0 \\
\iiint_{\mathcal{R}} (\vec{B} \cdot \hat{n}) \, dA = 0
\]

Faraday's Law of Induction: A changing magnetic field induces a current in a loop of wire.

Now

\[
\oint_{C} \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \iint_{S} \vec{B} \cdot \hat{n} \, dA
\]

electromotive force aka voltage

\[
\text{cur} \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\]

\[
\oint_{C} \vec{E} \cdot d\vec{r} = \iint_{S} (\text{curl} \vec{E}) \cdot \hat{n} \, dA
\]

and so
Unit check: \( F = qE \Rightarrow \vec{E} \) in \( \text{N/C} = \text{V/m} \) \( \Rightarrow \int \vec{E} \cdot d\vec{r} \) in \( \text{V} \), \( \text{m} \), \( \text{hrs} \).

(2) \( \vec{B} \) has units \( T = \text{Tesla} = \frac{\text{Vs}}{\text{m}^2} \) \( \Rightarrow \vec{n} = \text{unitless} \)

So \( \iiint \vec{B} \cdot \vec{n} \ dA \) is in \( \text{Vs} \) \( \Rightarrow \frac{\partial}{\partial t} \iiint (\vec{B} \cdot \vec{n}) \ dA \) is in \( \text{V} \).

Ampere's circuital law: Current in a wire or a changing electric field induces a magnetic field.

\[
\oint_C \vec{B} \cdot d\vec{r} = \mu_0 \iint (\vec{J} \cdot \vec{n}) \ dA + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \iint (\vec{E} \cdot \vec{n}) \ dA
\]

\[
\iint (\text{curl} \vec{B}) \cdot \vec{n} \ dA
\]

So \( \text{curl} \vec{B} = \mu_0 \text{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \)

\( \varepsilon_0 = \text{permittivity of free space} = \text{Farads/m} \)

\( \mu_0 = \text{permeability of free space} = \frac{N}{\text{A}^2} \)
Integral: 
\[ \int \omega = m \, \omega \] 

Time remains: 
(a) Multitude of integral theorems
(b) Multitude of integral theorems

Moral: Here are other applications of line and surface integrals, as well as the relations between them.

Check: 
\[ E = \frac{\partial E_{\text{now}}}{\partial t} \]
\[ B = \frac{\partial E_{\text{now}}}{\partial x} \]

\[ E = (0, c \cos(2\pi (ut - x)), 0) \]
\[ B = (0, 0, \cos(2\pi (ut - x))) \]

\[ \text{C = speed of light} \]

\[ \frac{1}{\sqrt{c^2 - \omega^2}} \]

\[ f = c^2 \pi^2 - \omega^2 \]