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TOPOLOGY FROM THE
DIFFERENTIABLE
VIEWPOINT

Revised Edition

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BASED ON NOTES BY DAVID W. WEAVER

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§1. SMOOTH MANIFOLDS
AND SMOOTH MAPS

First let us explain some of our terms. \( \mathbb{R}^k \) denotes the \( k \)-dimensional euclidean space; thus a point \( x \in \mathbb{R}^k \) is an \( k \)-tuple \( x = (x_1, \ldots, x_k) \) of real numbers.

Let \( U \subset \mathbb{R}^k \) and \( V \subset \mathbb{R}^l \) be open sets. A mapping \( f \) from \( U \) to \( V \) (written \( f : U \to V \)) is called smooth if all of the partial derivatives \( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k} \) exist and are continuous.

More generally let \( X \subset \mathbb{R}^k \) and \( Y \subset \mathbb{R}^l \) be arbitrary subsets of euclidean spaces. A map \( f : X \to Y \) is called smooth if for each \( x \in X \) there exist an open set \( U \subset \mathbb{R}^k \) containing \( x \) and a smooth mapping \( F : U \to \mathbb{R}^l \) that coincides with \( f \) throughout \( U \cap X \).

If \( f : X \to Y \) and \( g : Y \to Z \) are smooth, note that the composition \( g \circ f : X \to Z \) is also smooth. The identity map of any set \( X \) is automatically smooth.

**Definition.** A map \( f : X \to Y \) is called a diffeomorphism if \( f \) carries \( X \) homeomorphically onto \( Y \) and if both \( f \) and \( f^{-1} \) are smooth.

We can now indicate roughly what differential topology is about by saying that it studies those properties of a set \( X \subset \mathbb{R}^k \) which are invariant under diffeomorphism.

We do not, however, want to look at completely arbitrary sets \( X \). The following definition singles out a particularly attractive and useful class.

**Definition.** A subset \( M \subset \mathbb{R}^k \) is called a smooth manifold of dimension \( m \) if each \( x \in M \) has a neighborhood \( W \cap M \) that is diffeomorphic to an open subset \( U \) of the euclidean space \( \mathbb{R}^m \).

Any particular diffeomorphism \( g : U \to W \cap M \) is called a parameterization of the region \( W \cap M \). (The inverse diffeomorphism \( W \cap M \to U \) is called a system of coordinates on \( W \cap M \).)
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Figure 1. Parametrization of a region in M

Sometimes we will need to look at manifolds of dimension zero. By definition, $M$ is a manifold of dimension zero if each $x \in M$ has a neighborhood $W \cap M$ consisting of $x$ alone.

Examples. The unit sphere $S^2$, consisting of all $(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$ is a smooth manifold of dimension 2. In fact the diffeomorphism

$$(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2}),$$

for $x^2 + y^2 < 1$, parametrizes the region $z > 0$ of $S^2$. By interchanging the roles of $x, y, z$, and changing the signs of the variables, we obtain similar parametrizations of the regions $x > 0, y > 0, x < 0, y < 0$, and $z < 0$. Since these cover $S^2$, it follows that $S^2$ is a smooth manifold.

More generally the sphere $S^{n-1} \subset \mathbb{R}^n$ consisting of all $(x_1, \ldots, x_n)$ with $\sum x^2 = 1$ is a smooth manifold of dimension $n - 1$. For example $S^0 \subset \mathbb{R}^1$ is a manifold consisting of just two points.

A somewhat wilder example of a smooth manifold is given by the set of all $(x, y) \in \mathbb{R}^2$ with $x \neq 0$ and $y = \sin(1/x)$.

TANGENT SPACES AND DERIVATIVES

To define the notion of derivative $df_x$ for a smooth map $f : M \to N$ of smooth manifolds, we first associate with each $x \in M \subset \mathbb{R}^n$ a linear subspace $TM_x \subset \mathbb{R}^n$ of dimension $m$ called the tangent space of $M$ at $x$. Then $df_x$ will be a linear mapping from $TM_x$ to $TN_y$, where $y = f(x)$. Elements of the vector space $TM_x$ are called tangent vectors to $M$ at $x$.

Intuitively one thinks of the $m$-dimensional hyperplane in $\mathbb{R}^n$ which best approximates $M$ near $x$; then $TM_x$ is the hyperplane through the origin that is parallel to this. (Compare Figures 1 and 2.) Similarly one thinks of the nonhomogeneous linear mapping from the tangent hyperplane at $x$ to the tangent hyperplane at $y$ which best approximates $f$. Translating both hyperplanes to the origin, one obtains $df_x$.

Before giving the actual definition, we must study the special case of mappings between open sets. For any open set $U \subset \mathbb{R}^n$ the tangent space $TU$, is defined to be the entire vector space $\mathbb{R}^n$. For any smooth map $f : U \to V$ the derivative

$$df_x : \mathbb{R}^n \to \mathbb{R}^m$$

is defined by the formula

$$df_x(h) = \lim_{t \to 0} (f(x + th) - f(x))/t$$

for $x \in U$, $h \in \mathbb{R}^n$. Clearly $df_x(h)$ is a linear function of $h$. (In fact $df_x$ is just that linear mapping which corresponds to the $l \times k$ matrix $(df_x/\partial x_i)_j$ of first partial derivatives, evaluated at $x$.)

Here are two fundamental properties of the derivative operation:

1. (Chain rule). If $f : U \to V$ and $g : V \to W$ are smooth maps, with $f(x) = y$, then

$$d(g \circ f)_x = dg_y \circ df_x.$$

In other words, to every commutative triangle

$$\begin{array}{ccc}
V & \xrightarrow{g} & W \\
\xrightarrow{f} & & \\
U & \xrightarrow{g \circ f} & W
\end{array}$$

of smooth maps between open subsets of $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^n$, there corresponds a commutative triangle of linear maps

$$\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{d(g \circ f)_x} & \mathbb{R}^n \\
\xrightarrow{d(g)_y} & & \\
\mathbb{R}^n & \xrightarrow{df_x} & \mathbb{R}^m
\end{array}$$

2. If $I$ is the identity map of $U$, then $dI_x$ is the identity map of $\mathbb{R}^n$. More generally, if $U \subset U'$ are open sets and

$$i : U \to U'$$

is the inclusion, then $di_x$ is the identity map of $\mathbb{R}^n$.
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is the inclusion map, then again $d_i$ is the identity map of $R^i$.

Note also:

3. If $L : R^n \to R^i$ is a linear mapping, then $dL = L$.

As a simple application of the two properties one has the following:

**Assertion.** If $f$ is a diffeomorphism between open sets $U \subset R^n$ and $V \subset R^i$, then $k$ must equal $l$, and the linear mapping

$$df : R^n \to R^i$$

must be nonsingular.

**Proof.** The composition $f^{-1} \circ f$ is the identity map of $U$; hence $d(f^{-1} \circ df)$ is the identity map of $R^n$. Similarly $df \circ d(f^{-1})$ is the identity map of $R^i$. Thus $df$ has a two-sided inverse, and it follows that $k = l$.

A partial converse to this assertion is valid. Let $f : U \to R^n$ be a smooth map, with $U$ open in $R^n$.

**Inverse Function Theorem.** If the derivative $df : R^n \to R^i$ is nonsingular, then $f$ maps any sufficiently small open set $U'$ about $x$ diffeomorphically onto an open set $f(U')$.

(See Apostol [2, p. 144] or Dieudonné [7, p. 268].)

Note that $f$ may not be one-one in the large, even if every $df$, is nonsingular. (An instructive example is provided by the exponential mapping of the complex plane into itself.)

Now let us define the tangent space $TM_x$ for an arbitrary smooth manifold $M \subset R^n$. Choose a parametrization

$$g : U \to M \subset R^n$$

of a neighborhood $g(U)$ of $x$ in $M$, with $g(u) = x$. Here $U$ is an open subset of $R^n$. Think of $g$ as a mapping from $U$ to $R^n$, so that the derivative

$$dg : R^n \to R^n$$

is defined. Set $TM_x$ equal to the image $dg(R^n)$ of $dg$. (Compare Figure 1.)

We must prove that this construction does not depend on the particular choice of parametrization $g$. Let $h : V \to M \subset R^n$ be another parametrization of a neighborhood $h(V)$ of $x$ in $M$, and let $v = h^{-1}(x)$. Then $h^{-1} \circ g$ maps some neighborhood $U_i$ of $u$ diffeomorphically onto a neighborhood $V_i$ of $v$. The commutative diagram of smooth maps

![Tangent spaces](image)

gives rise to a commutative diagram of linear maps

$$dg \circ R^n \cong dh \circ R^n$$

and it follows immediately that

$$\text{Image}(dg) = \text{Image}(dh).$$

Thus $TM_x$ is well defined.

**Proof that $TM_x$ is an m-dimensional vector space.** Since

$$g^{-1} : g(U) \to U$$

is a smooth mapping, we can choose an open set $W$ containing $x$ and a smooth map $F : W \to R^n$ that coincides with $g^{-1}$ on $W \cap g(U)$. Setting $U_0 = g^{-1}(W \cap g(U))$, we have the commutative diagram

![Diagram](image)

and therefore

$$dg : R^n \to R^n$$

This diagram clearly implies that $dg$ has rank $m$, and hence that its image $TM_x$ has dimension $m$.

Now consider two smooth manifolds, $M \subset R^n$ and $N \subset R^l$, and a
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smooth map

\[ f : M \rightarrow N \]

with \( f(x) = y \). The derivative

\[ df_x : TM_x \rightarrow TN_y \]

is defined as follows. Since \( f \) is smooth there exist an open set \( W \) containing \( x \) and a smooth map

\[ F : W \rightarrow \mathbb{R}^i \]

that coincides with \( f \) on \( W \cap M \). Define \( df_x(v) \) to be equal to \( dF_x(v) \) for all \( v \in TM_x \).

To justify this definition we must prove that \( dF_x(v) \) belongs to \( TN_y \) and that it does not depend on the particular choice of \( F \).

Choose parametrizations

\[ g : U \rightarrow M \subset \mathbb{R}^k \text{ and } h : V \rightarrow N \subset \mathbb{R}^i \]

for neighborhoods \( g(U) \) of \( x \) and \( h(V) \) of \( y \). Replacing \( U \) by a smaller set if necessary, we may assume that \( g(U) \subset W \) and that \( f \) maps \( g(U) \) into \( h(V) \). It follows that

\[ h^{-1} \circ f \circ g : U \rightarrow V \]

is a well-defined smooth mapping.

Consider the commutative diagram

\[ \begin{array}{ccc}
W & \xrightarrow{F} & \mathbb{R}^i \\
\uparrow & & \uparrow \\
U & \xrightarrow{g} & \mathbb{R}^k \\
\downarrow h^{-1} \circ f \circ g & & \\
V & \xrightarrow{h} & \mathbb{R}^i
\end{array} \]

of smooth mappings between open sets. Taking derivatives, we obtain a commutative diagram of linear mappings

\[ \begin{array}{ccc}
\mathbb{R}^k & \xrightarrow{dF_x} & \mathbb{R}^i \\
\uparrow dg_x & & \uparrow dh_x \\
\mathbb{R}^k & \xrightarrow{d(h^{-1} \circ f \circ g)_x} & \mathbb{R}^i
\end{array} \]

where \( u = g^{-1}(x), v = h^{-1}(y) \).

It follows immediately that \( dF_x \) carries \( TM_x = \text{Image} \ (dg_x) \) into \( TN_y = \text{Image} \ (dh_x) \). Furthermore the resulting map \( df_x \) does not depend on the particular choice of \( F \), for we can obtain the same linear transformation by going around the bottom of the diagram. That is:

\[ df_x = dh_x \circ d(h^{-1} \circ f \circ g)_x \circ (dg_x)^{-1}. \]

This completes the proof that

\[ df_x : TM_x \rightarrow TN_y \]

is a well-defined linear mapping.

As before, the derivative operation has two fundamental properties:

1. (Chain rule). If \( f : M \rightarrow N \) and \( g : N \rightarrow P \) are smooth, with \( f(x) = y \), then

\[ d(g \circ f)_x = dg_y \circ df_x. \]

2. If \( I \) is the identity map of \( M \), then \( di_x \) is the identity map of \( TM_x \).

More generally, if \( M \subset N \) with inclusion map \( i \), then \( TM_x \subset TN_x \) with inclusion map \( di_x \). (Compare Figure 2.)

![Figure 2. The tangent space of a submanifold](image)

The proofs are straightforward.

As before, these two properties lead to the following:

**Assertion.** If \( f : M \rightarrow N \) is a diffeomorphism, then \( df_x : TM_x \rightarrow TN_y \) is an isomorphism of vector spaces. In particular the dimension of \( M \) must be equal to the dimension of \( N \).

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### REGULAR VALUES

Let \( f : M \rightarrow N \) be a smooth map between manifolds of the same dimension. We say that \( x \in M \) is a regular point of \( f \) if the derivative

* This restriction will be removed in §2.
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$df_x$ is nonsingular. In this case it follows from the inverse function theorem that $f$ maps a neighborhood of $x$ in $M$ diffeomorphically onto an open set in $N$. The point $y \in N$ is called a regular value if $f^{-1}(y)$ contains only regular points.

If $df_x$ is singular, then $x$ is called a critical point of $f$, and the image $f(x)$ is called a critical value. Thus each $y \in N$ is either a critical value or a regular value according as $f^{-1}(y)$ does or does not contain a critical point.

Observe that if $M$ is compact and $y \in N$ is a regular value, then $f^{-1}(y)$ is a finite set (possibly empty). For $f^{-1}(y)$ is in any case compact, being a closed subset of the compact space $M$; and $f^{-1}(y)$ is discrete, since $f$ is one-one in a neighborhood of each $x \in f^{-1}(y)$.

For a smooth $f : M \to N$, with $M$ compact, and a regular value $y \in N$, we define $\#f^{-1}(y)$ to be the number of points in $f^{-1}(y)$. The first observation to be made about $\#f^{-1}(y)$ is that it is locally constant as a function of $y$ (where $y$ ranges only through regular values!). I.e., there is a neighborhood $V \subset N$ of $y$ such that $\#f^{-1}(y') = \#f^{-1}(y)$ for any $y' \in V$. [Let $x_1, \ldots, x_s$ be the points of $f^{-1}(y)$, and choose pairwise disjoint neighborhoods $U_1, \ldots, U_s$ of these which are mapped diffeomorphically onto neighborhoods $V_1, \ldots, V_s$ in $N$. We may then take

$$V = V_1 \cap V_2 \cap \cdots \cap V_s - f(M - U_1 - \cdots - U_s).$$]

THE FUNDAMENTAL THEOREM OF ALGEBRA

As an application of these notions, we prove the fundamental theorem of algebra: every nonconstant complex polynomial $P(z)$ must have a zero.

For the proof it is first necessary to pass from the plane of complex numbers to a compact manifold. Consider the unit sphere $S^2 \subset \mathbb{R}^3$ and the stereographic projection

$$h_+: S^2 - \{(0, 0, 1)\} \to \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$$

from the "north pole" $(0, 0, 1)$ of $S^2$. (See Figure 3.) We will identify $\mathbb{R}^2 \times 0$ with the plane of complex numbers. The polynomial map $P$ from $\mathbb{R}^2 \times 0$ to itself corresponds to a map $f$ from $S^2$ to itself; where

$$f(x) = h_+^{-1}Ph_+(x) \quad \text{for} \quad x \neq (0, 0, 1)$$

$$f(0, 0, 1) = (0, 0, 1).$$

It is well known that this resulting map $f$ is smooth, even in a neighbor-

![Figure 3. Stereographic projection](image-url)