Thm: $G$ a finite group. Then $\exists$ a Galois extension $K$ of $\mathbb{C}(t)$ with $\text{Gal}(K/\mathbb{C}(t))$.

Last time: Given an irreducible curve $V \subseteq \mathbb{C}^2$, a poly fn $h \in \mathbb{C}[V]$ (e.g. proj to the $x$-axis) get that $K = \mathbb{C}(V)$ is a finite extension of $\mathbb{C}(t)$.

Plan: 1) Given $G$ find a curve $V$ in $\mathbb{P}_C^2$ on which $G$ acts via symmetries, so that $V/G = \mathbb{P}_C^1 = \mathbb{O}$

2) Each $\sigma \in G$, thought of a sym of $V$, gives an auto of $K = \mathbb{C}(V)$, via

\[ \sigma^*(f) = f \circ \sigma^{-1} \text{ where } f: V \rightarrow \mathbb{P}_C^1. \]

[Aside: Check about group actions]
3. \( K_G = \mathcal{C}(V)_G = \mathcal{C}(V/G) = \mathcal{C}(P^r_2) = \mathcal{C}(t) \).

**Toy ex:**  
\[ V = \begin{array}{ccc} 2 & 1 \\ \downarrow & & \downarrow \\ 1 & 2 & 1 \\ \downarrow & & \downarrow \\ 2 & 1 \end{array} \]

\[ V/G = \{ \text{half of one side} \} \]

\( G = D_4 \)

Sadly, don't time to prove the whole thing as need a third perspective.

\[ \{ \text{Alg. curves over } \mathbb{C} \} \leftrightarrow \{ \text{finite extensions of } \mathbb{C}(t) \} \]

Complex analysis = \{ Riemann surfaces \}

Also need some topology of covering spaces.

Instead, I'll do an example with \( G = S_3 \).
Given a finite group $G$, let's make it act on some geometric object.

**Def:** Let $S$ be a generating set for $G$.

The Cayley Graph $\Gamma(G, S)$ has:

1. a vertex $V_g$ for each $g \in G$.
2. an edge labeled $s$ from $V_g$ to $V_{gs}$ for each $g \in G$ and $s \in S$.

**Ex:** $S_3 = \{1, (12), (13), (23), (123), (132)\}$

$S = \{a = (12), b = (123)\}$
Q: Is $abab^{-1}ab$? 

\[ A, (12) = a. \]

For any $(G, S)$, the Cayley graph is very symmetric. In particular, $G$ acts on $\Gamma$ via:

\[ g \cdot V_h = V_{gh} \quad \text{(which doesn't break)} \]

In our example:

We have:

- $a$ rotates by $\pi$.
- $b$ rotates by $2\pi/3$. 

Comment on Expanders/Geometric Group Theory!
What is $\Gamma/\Gamma_0$? 

Now we want $\Gamma$ to act on a surface, so "thicken" $\Gamma/\Gamma_0$ to 

and corresponding $\Gamma$ to 

Now for each circle boundary component, add a disc.
So \( \Gamma/\mathbb{G} \) becomes

\[
X = \begin{array}{c}
\infty
\end{array}
= \mathbb{P}^1
\]

And \( \mathbb{G} \) becomes

\[
Y = \begin{array}{c}
\text{as well.}
\end{array}
\]

The action of \( \mathbb{G} \) on \( \Gamma \) gives an action of \( \mathbb{G} \) on \( Y \).

- \( (13) \) rotates by \( \pi \)
- \( (23) \) rotates by \( \pi \)
- \( a \) rotates by \( \pi \)
- \( b \) rotates by \( 2\pi/3 \)

\[
[S_3 = \text{orientable isom of the bipyramid}]
\]
What is $p : Y \rightarrow X$ like?

First, notice $\pi : \Gamma \rightarrow \Gamma / G$ is locally 1-1 (a homeomorphism). The same is true for $p : Y \rightarrow X$, except at the 8 points the are fixed by some elt of $G$ (these are the centers of the added discs).

At these pts looks like

\[ \begin{array}{c}
\pi \\
\downarrow \\
\mathbb{Z}^2 \\
\end{array} \quad \text{or} \quad \begin{array}{c}
2\pi/3 \\
\downarrow \\
\mathbb{Z}^3 \\
\end{array} \]
So, locally, \( p \) looks like a polynomial.

Now, we invoke the Riemann existence theorem to turn this into a honest rational map \( P^1 \to P^1 \). This will give an extension \( K/C(t) \) with Galois group \( S_3 \).

Note: The construction of \( p: Y \to X \) from \( T(G, S) \) is general. Let's the R. E. T. that is hard...