Last time:

\[ K/F \text{ is a field extension means } F \subseteq K, \]

\[ [K:F] = \text{dim of } K \text{ as an } F \text{-vector space} \]

\[ p(x) \text{ is a poly in } F[x], \text{ form a field} \]

\[ K = F[x] / \langle p(x) \rangle \quad \longleftrightarrow \quad \text{polys in } F[x] \]

\[ \text{of deg } < \text{deg } p \]

\[ \Rightarrow [K:F] = \text{deg } p. \]

Think of K as adding a root of p to F.

Explicitly, let \( \theta = x + \langle p(x) \rangle \). Then

\[ p(\theta) = p(x) + \langle p(x) \rangle = 0 \text{ in } K. \]

An F-basis of K is \( 1, \theta, \theta^2, \ldots, \theta^n \) where \( n = (\text{deg } p) - 1. \)

Ex: \( F = \mathbb{R}, \ p = x^2 + 1 \)

\[ K = \mathbb{R}[x] / \langle x^2 + 1 \rangle \quad \cong \quad \mathbb{C} \]

\[ 1 \quad 1 \]

\[ \theta \quad i \text{ (or } -i) \]
**Notation:** \( \alpha_1, \ldots, \alpha_n \in K \) with \( F \subseteq K \) then \( F(\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{field gen by els of } F \text{ and the } \alpha_i. \) [i.e. the smallest subfield containing all of them.]

**Ex:** \( \mathbb{Q}(i), \mathbb{Q}(\sqrt{2}, \sqrt{5}). \) Here the large field is \( \mathbb{C}. \)

**Simple Extension:** \( K = F(\alpha) \) for some \( \alpha \in F. \)

↑ primitive elt.

**Ex:** \( \mathbb{Q}(\sqrt{2}, \sqrt{5}) / \mathbb{Q} \) is simple as \( \mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\alpha = \sqrt{2} + \sqrt{5}) \)

since \( \sqrt{2} = \frac{1}{6}(\alpha^3 - 11\alpha) \)

**Will show:** Any \( K/F \) with \([K:F] < \infty \) and \( \text{ch}(F) = 0 \) is primitive.

**Thm:** \( p(x) \in F[x] \) indep. Suppose \( K \) is a simple ext. of \( F \) w/ primitive elt \( \alpha. \) If \( p(\alpha) = 0, \) then

\[ L = F[x] / (p(x)) \cong K \]

**Proof:** Consider \( \phi : L \rightarrow K \) given by

\[ g(x) + (p(x)) \mapsto g(\alpha) \]

Makes sense because \( f(\alpha) = 0 \) if \( f \in (p(x)) \) and...
is a ring hom by the basic ring axioms.

Lemma: \( \phi: L \to K \) a ring hom of fields.

Then either \( \phi(L) = 0 \) or \( \phi \) is 1-1.

Reason: \( \ker(\phi) = \{ \phi(x) = 0 \mid x \in L \} \) is an ideal

hence either 0 or L as every elt of \( L \setminus \ker(\phi) \) is a unit.

Our \( \phi \) is not trivial as \( \phi/_{\text{const}} \) is an isom to \( F \)

so its 1-1. Moreover, \( \phi \) is onto as its image
contains \( F \) and \( \alpha \). So \( \phi \) is an isom.

Thm: Suppose \( K = F(\alpha) \) with \( [K: F] = n < \infty \).

Then \( \exists \) a monic poly \( p(x) \in F[x] \) with
\( p(\alpha) = 0 \). Thus \( K \cong F[x]/(p(x)) \).

Ex: \( \mathbb{Q}(\sqrt{2}) = \mathbb{Q}[x]/(x^2 - 2) \)

\( \mathbb{Q}(\alpha = \sqrt{2} + \sqrt{3}) = \mathbb{Q}[x]/(x^4 - 14x^2 + 9) \)
\textbf{Pf:} As \( \dim K \) as a \( F \)-vector space is \( n \),

\[ 1, x, x^2, \ldots, x^n \]

must be linearly dependent, i.e., \( \exists a \in F \) with

\[ a_0, 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0 \]

do take \( p(x) = a_0 + a_1 x + \ldots + a_n x^n \in F[x] \).

If \( p(x) \) is real, replace it with an irreducible factor which also has \( \alpha \) as a root.

\textit{Note: A posteriori,} \( p \) must be irreducible, as

\[ \left[ \frac{F[x]}{(p(x))} : F \right] = \deg p \]

\textbf{Ex:} \( \mathbb{Q}(\sqrt{2}, \sqrt{5}) \) has \( \mathbb{Q} \)-basis \( 1, \sqrt{2}, \sqrt{5}, \sqrt{10} \)

Above things follow easily by computing in this basis.

\[ \alpha = \sqrt{2} + \sqrt{5} \]

\[ \alpha^2 = 7 + 2\sqrt{10} \]

\[ \alpha^3 = 17\sqrt{2} + 11\sqrt{5} \]

\[ \alpha^4 = 89 + 28\sqrt{10} \]
What if \([F(\alpha) : F] = \infty\)? Then
\[p(\alpha) \neq 0 \text{ for every } p \in F[x].\] 
\[\exists \alpha = \pi, F = \mathbb{Q}\]
(Otherwise \(F(\alpha) \cong F[x]/q(x)\) for some irreducible factor \(q\) of \(p\).)

Consider the field of all \(x\)'s in \(x\) over \(F:\)

\[F(x) = \frac{\text{true field}}{\text{true field}} = \left\{ \frac{p(x)}{q(x)} \mid p, q \in F[x], q \neq 0 \right\} / \sim\]

Then
\[
\phi: F(x) \longrightarrow F(\alpha)
\]

\[
\frac{p(x)}{q(x)} \longrightarrow \frac{p(\alpha)}{q(\alpha)}
\]

makes sense because \(f(\alpha) = 0 \Rightarrow f = 0 \) in \(F[x]\)

As before it's an isom. So \(F(\alpha) \cong F(x)\).

Cor: \(\mathbb{Q}(\pi), \mathbb{Q}(e), \mathbb{Q}(\ln 2)\)

are all isomorphic fields (\(\cong \mathbb{Q}(x)\)).