Lecture 12: Limitations of Straightedge and Compass.

Given a ruler and compass, what can you construct?

2. Perpendicular Bisectors:

3. Parallel lines:

4. Divide a segment in half

4. Double the area of a square

5. Make a regular 17-gon. (Or a 65537-gon!)
Things you can't do

1. Trisect an angle

2. Given a circle, build a square of the same area.

Rules: (Goes back to the ancient Greeks)

A. Given two points can draw
   1. The line joining them
   2. The circle centered at one point, passing through the other

B. Find the points of intersection of lines and circles

Note: Can't measure things.

Starting Setup: Two points:

Consider the constructable numbers:

\[ C = \{ d \in \mathbb{R} \mid \text{starting with } 1 \text{ can construct two points } x, y \text{ with } d(x, y) = \pm d \} \]
Observations:

0. Impose a Cartesian coordinate system with one pt at 0 and the other at (1,0).

Prop: \( p \in C \) iff can construct the point \((d,0)\).

Pf: \((\Leftarrow)\) Clear

\((\Rightarrow)\) Given 3 points, can construct a circle centered about one of radii the dist between the others [Or use a modern compass.]

Prop: A point \( p \in \mathbb{R}^2 \) is constructible \(\iff\) both coords are in \( C \).
Of: \((\Rightarrow)\) 

2. \(C\) is field.

3. \(c\) if \(a \in C\),
   
   so is \(\sqrt{a}\).
Thm: If \( a \in \mathbb{C} \), then \([\mathbb{Q}(a): \mathbb{Q}] = 2^n\)

Cor: Can't trisect angles

Proof: If so, then we can construct the point on the unit circle with angle \( 20^\circ = \pi/9 \).

Thus \( c = \cos \pi/9 \) is in \( \mathbb{C} \). By the triple angle formula,

\[
\frac{1}{2} = \cos \pi/3 = 4 \cos^3 \pi/9 - 3 \cos \pi/9
\]

\[
\Rightarrow 8c^3 - 6c - 1 = 0
\]

Since \( 8x^3 - 6x - 1 \) is irreducible in \( \mathbb{Z}[x] \) (and hence in \( \mathbb{Q}[x] \)) (reduce mod 3), \([\mathbb{Q}(c): \mathbb{Q}] = 3\). But then \( c \) can't be in \( \mathbb{C} \).
Can’t square the circle

\[ \sqrt{\pi} \rightarrow \sqrt{\pi} \]

Area = \pi

Would imply that \( \sqrt{\pi} \in \mathbb{C} \Rightarrow \pi \in \mathbb{C} \)

\( \Rightarrow [Q(\pi) : \mathbb{Q}] < \infty \), but \( \pi \) is transcendental.

Proof of \( \text{thm} \): Fix \( a \in \mathbb{C} \). Then \( (a, 0) \) is constructible in a series of steps. Let \( F_n = Q(\cos \text{ of first } n \text{ points constructed}) \).

Then \( F_{n+1} = F_n (a_{n+1}, b_{n+1}) \)

\[ \frac{\text{cos of } P_{n+1}}{\text{cos of } P_n} \]

Now \( P_{n+1} \) is found by intersecting lines and circles, and the signs for such have coeffs in \( F_n \).
Intersecting two lines:

linear equations \( \Rightarrow F_{n+1} = F_n \)

Intersecting a line with a circle: Degree 2 extension

Intersecting two circles reduces to previous case.

Details next time.