Last time: \( K/F \) field extension, \( \alpha \in K \) in algebraic over \( F \) if \( \exists f(x) \in F[x] \) with \( f(\alpha) = 0 \).

\( K/F \) is algebraic if every elt of \( K \) is alg. over \( F \).

Thm: \([K:F] < \infty \Rightarrow K/F \) is algebraic,

Thm: \( \alpha, \beta \in K \) algebraic, then \( F(\alpha, \beta)/F \) is alg.

Cor: \( \mathbb{Q} = \{ \alpha \in \mathbb{C} \mid \alpha \text{ alg over } \mathbb{Q} \} \) is a field.

Thm: \( F \leq K \leq L \) fields. Then \([L:F] = [L:K][K:F]\)

Pf: If \([L:F] < \infty \), so is \([L:K]\) and \([K:F]\).

So assume \([L:K]\) and \([K:F]\) are finite.

\( L \supset \{ \beta_1, \ldots, \beta_n \} \) a \( K \)-basis for \( L \).

\( K \supset \{ \alpha_1, \ldots, \alpha_m \} \) a \( F \)-basis for \( K \).

\( F \) Then \( \{ \delta_{ij} = \alpha_i \beta_j \} \) are m.n elts which \( F \)-span \( L \).
Claim: \{ \gamma_i \} are linearly indep, and so form a basis. Hence \([L:K] = nm\), as needed.

Suppose \(a_{ij} \in F\) with \(\sum_{i,j} a_{ij} \delta_{ij} = 0\).

Then \(\sum_j (\sum_i a_{ij} \alpha_i) \beta_j = 0\) and so as the \(\beta_j\) are \(K\)-linearly indep, must have each \(\sum_i a_{ij} \alpha_i = 0\). As \(\alpha_i\) are \(F\)-lin indep, get all \(a_{ij} = 0\). So \(\{ \delta_{ij} \}\) are \(F\)-lin indep.

Finite Extension: \(K/F\) with \([K:F] < \infty\).

Cor: \(F \subseteq K \subseteq L\). \(Cl F/K\) and \(K/F\) are finite, so is \(L/F\).

Thm: \(F \subseteq K \subseteq L\). \(Cl L/K\) and \(K/F\) are algebraic, so is \(L/F\).
Proof: A given $\beta \in L$ is a root of some
\[ p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in K[x] \]
Consider $F \leq F(\alpha_0) \leq F(\alpha_1, \alpha_0) \leq \ldots$
\[ \leq F(\alpha_0, \alpha_1, \ldots, \alpha_n) \leq F(\alpha_0, \alpha_1, \ldots, \alpha_n, \beta) = M \]
These are all finite extensions $\Rightarrow M/F$ is finite
$\Rightarrow \beta$ is alg. over $F$. So $L/F$ is algebraic. $
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Consequences of $L = [L:F] = [L:K][K:F]$ \\
\[ L, K, F \]

Q: Does one of $Q(\sqrt{2})$ and $Q(3\sqrt{2})$
contain the other? (Both $\subseteq R$)
A: No since $[Q(\sqrt{2}) : Q] = 2$ and $[Q(3\sqrt{2}) : Q] = 3$
and if $Q \subseteq Q(\sqrt{2}) \subseteq Q(3\sqrt{2})$ then one has
$2 + 3$ which is silly.

Composite of fields $K_1, K_2 \subseteq L$ is the smallest
such field which contains both, and is
denoted $K_1K_2$. 
Ex: \( \mathbb{Q}(\sqrt{2}) \mathbb{Q}(3\sqrt{2}) = \mathbb{Q}(6\sqrt{2}) \)

Reason 1: \( \mathbb{Q}(6\sqrt{2}) \) contains \( \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(3\sqrt{2}) \)
and \( \sqrt{2}/3\sqrt{2} = 2^{1/2} \cdot 2^{-1/3} = 2^{-1/6} = 6\sqrt{2} \).

Reason 2: Any field containing \( \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(3\sqrt{2}) \) must have \( [K: \mathbb{Q}] \) divisible by 2 + 3
\( \Rightarrow [K: \mathbb{Q}] \geq 6 \). As \( [\mathbb{Q}(\sqrt{2}): \mathbb{Q}] = 2 \),
it must be the compositum.

Thm: \( F \subseteq K_1, K_2 \subseteq L \), with \( [K_i : F] < \infty \)
\[ [K_1 K_2 : F] \leq [K_1 : F][K_2 : F] \]

Note: Clearly both \( K_i/F \) are simple.

Qf: Let \( \{ \alpha_i \} \) be an \( F \)-basis for \( K_1 \), with \( \alpha_1 = 1 \)
\( \{ \beta_j \} \) be an \( F \)-basis for \( K_2 \), with \( \beta_1 = 1 \)

Claim: \( K_1 K_2 = \{ \sum a_{ij} \alpha_i \beta_j \mid a_{ij} \in F \} = K \)

Clearly \( K_i \subseteq K \subseteq K_1 K_2 \) so the issue is whether \( K \) is a subfield.
It's closed under $+$, and also $\times$ since

$$(\alpha_i \beta_j, \alpha_k \beta_l) = (\alpha_i \alpha_k)(\beta_j \beta_l) =$$

$$= (\sum a_i \alpha_i)(\sum b_j \beta_j) = \sum a_i b_j \alpha_i \beta_j$$

What about multiplicative inverses?

Fix $\gamma \in K$. Consider $T : K \rightarrow K$, which

$$\delta \mapsto \delta \gamma$$

is an $F$-linear transformation. As $L$ is an intermediate field, we have $\ker(T) = \{0\} \Rightarrow T$ is onto as $[K:F] < \infty$.

In particular, there exists $\delta \in K$ with $T(\delta) = 1$, i.e.

$$\delta \gamma = 1 \Rightarrow \gamma^{-1} = \delta \in K$$. So $K$ is a subfield and hence $= K_1 \cup K_2$. 

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