Lecture 37:

Topology of plane curves in $\mathbb{P}_C^2$:

$V = V(f)$ where $f$ = homog. poly in $\mathbb{C}[x, y, z]$.
Assume $V$ is smooth and irreducible.

Examples we've seen:

1. $f$ linear, i.e. $V$ is a line. By HW, all are the same, so can focus on
   
   $V = V(y) = (y\text{-axis} + pt \text{ at } \infty) = \mathbb{P}_C^1 = \mathbb{P}$

2. $f$ quad, i.e. $V$ = conic.
   
   $V = \mathbb{P}_C^1 = \mathbb{P}$

3. $f$ cubic, i.e. $V$ = elliptic curve which has a group law.
   
In general, $V$ is a compact orientable surface and is one of

$[g \text{ is called the genus of } V]$  

$g = 0, 1, 2, 3, ...$
While this is over $\mathbb{C}$, there are important consequences even when $k = \mathbb{Q}$.

**Ex:**

**F.T.T.** When $n \geq 3$,

$$\left\{ V_{p^n} (x^n + y^n - z^n) \right\} = \emptyset$$

Suppose $f \in \mathbb{Q}[x,y,z]$ homogenous. Consider

$$\left( V_{\mathbb{Q}} = V_{p^n_{\mathbb{Q}}} (f) \right) \subseteq \left( V_{\mathbb{C}} = V_{p^n_{\mathbb{C}}} (f) \right)$$

How many points $V_{\mathbb{Q}}$ has depends on the genus of $V_{\mathbb{C}}$. 
<table>
<thead>
<tr>
<th>genus</th>
<th>( \mathcal{V}_g )</th>
<th>Symmetries of ( \mathcal{V}_g )</th>
<th>Geometry of ( \mathcal{V}_g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \mathbb{P}^1 ) or ( \emptyset ) ( x^2 + y^2 - z^2 ) ( x^2 + y^2 - 3z^2 )</td>
<td>( \text{PGL}_2 \mathbb{C} = \frac{az + b}{cz + d} )</td>
<td>round sphere with unique shape</td>
</tr>
<tr>
<td>1</td>
<td>( \mathcal{V}_g ) is a finitely gen gp (Mordell-Weil) ( \exists p_i \in \mathcal{V}_g ) s.t. ( \mathcal{V}_g = { n_1 p_1 + \ldots + n_k p_k</td>
<td>n_i \in \mathbb{Z} } )</td>
<td>trans. by group sets + finite gp</td>
</tr>
<tr>
<td>2</td>
<td>Falting's Thm (1980s) ( \mathcal{V}_g ) is finite</td>
<td>finite</td>
<td>Hyperbolic geom. ( 3g - 3 = \text{dim moduli space} )</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \emptyset ) (Almost gives FLT)</td>
<td></td>
<td></td>
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Goal:

Thm: $G$ a finite gp. Then $\exists$ a Galois extension $K/E(t)$ with group $G$.

So, need to associate a field with a variety somehow...

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$V$ affine alg. variety $\subseteq k^n$.

$k[V] = \{ f: V \to k \mid f = \text{rest. of a poly} \}$

$= k[x_0, \ldots, x_n]/\mathfrak{I}(V)$

If $V$ is irreducible, then $k[V]$ is an integral domain.

Def: The function field of an irreducible variety $V$, denoted $k(V)$, is the
the field of fractions of \( k[V] \).

An elt of \( k(V) \) is call a rational function and is

\[
f = \frac{g}{h} \quad \text{for poly}\ g, h \in k[x_1, \ldots, x_n]
\]

Ex: \( k = \mathbb{C} \) and \( V = \mathbb{C} \). Then

\[C[V] = C[t]\] and so

\[C(V) = \text{rat'l fns in } t = C(t)\] [Notice connection to Goal Thm]

\[
f = c \frac{(t-a_1) \cdots (t-a_k)}{(t-b_1) \cdots (t-b_2)} \quad \text{no } a_i = b_j, \ c \in \mathbb{C}.
\]

Not quite a function \( f: V \to \mathbb{C} \) as not defined at \( b_i \).

Def: \( f \in k(V) \) is regular at \( p \in V \) if there is an expression \( f = \frac{g}{h} \) where \( h(p) \neq 0 \).

Set \( \text{dom}(f) = \{ p \in V \mid f \text{ reg at } p \} \).
Ex: $k = V = C, \#C$ with $f \in C(V)$ as above. Then
\[
\text{dom}(f) = C \setminus \{b_0, \ldots, b_{c-1}\}.
\]

Ex: $V = \mathbb{V}(xw - yz) \subseteq k^4$

Consider $f = \frac{x}{y} \in k(V)$. As $xw = yz$ in $k[V]$, another valid description of $f$ is $f = \frac{z}{w}$. Thus
\[
\text{dom}(f) = \{\text{all pts of } V \text{ with } y \neq 0 \text{ or } w \neq 0\}
\]

Underlying point: $k[V]$ is not a U.F.D.
Now focus on
\[ V = V(p) \subseteq \mathbb{C}^2 \]
a smooth irreducible plane curve.

Prop: If \( f \in C(V) \) then
\[ \text{dom}(f) = V \setminus \{ \text{finite set of pts} \} \]
\[ \text{pf}: f = \frac{g}{h} \text{ for } g, h \in C[V]. \text{ Need } h(p) = 0 \text{ for only finitely many points } p. \]

Equivalently, \( V' = V(h, p) = \text{finite set}. \)

Two approaches

1. Dimension of a variety

2. Can't happen.