1. Let $F$ be a field. Consider the ring $R = F[[t]]$ of formal power series in $t$, namely things of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 t + a_2 t^2 + \cdots$$

where $a_n \in F$.

Here “formal” means the above “sum” is really just an infinite list of elements of $F$; there's no notion of convergence involved. Elements of $R$ are added term by term, and multiplication is as if they were polynomials. More precisely

$$\sum_{n=0}^{\infty} a_n t^n \times \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) t^n$$

It is clear that $R$ is a commutative ring with unit.

(a) Prove that $\alpha$ in $R$ is a unit if and only if the constant term $a_0 \neq 0$. (Example: The inverse of $1 - t$ is $1 + t + t^2 + t^3 + t^4 + \cdots$)

(b) Prove that $R$ is a Euclidean domain with respect to the norm $N(\alpha) = n$ if $a_n$ is the first term of $\alpha$ that is non-zero. (If $F = \mathbb{C}$ and the power series converges near $t = 0$, then this norm is just the order of zero of the corresponding function at 0.)

(c) In the polynomial ring $R[x]$, prove that $x^n - t$ is irreducible.

2. Let $R = \mathbb{Z}[i]$.

(a) Prove that $R/(1 + i)$ is a field of order 2.

(b) Let $\pi \in R$ be irreducible. Consider the ideals $I_n = (\pi^n)$. Prove that $R/(\pi) \cong I_n/I_{n+1}$ as additive abelian groups. Hint: the isomorphism is multiplication by $\pi^n$.

(c) Again for irreducible $\pi$, prove that $|R/(\pi^n)| = |R/(\pi)|^n$. Here $| \cdot |$ denotes the number of elements in a finite set. (This is a key step in proving that for any $\pi \in R$ that $|R/(\pi)| = N(\pi) = |\pi|^2$.)

(d) For $\pi = 1 + i$, are $R/(\pi^3)$ and $\mathbb{Z}/8\mathbb{Z}$ isomorphic as rings?
5. Section 13.2, #20.