Discussion — Tuesday, October 12th

Subject: Integrating vector fields.

1. Consider the curve $C$ and vector field $\mathbf{F}$ shown below.

(a) Without parameterizing $C$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Hint: Use one of the reformulations of $\int_C \mathbf{F} \cdot d\mathbf{r}$ from Monday’s lecture.)

(b) Find a parameterization of $C$ and use it to check your answer in (a) by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$ explicitly.

2. Consider the vector field $\mathbf{F} = (y, 0)$ on $\mathbb{R}^2$.

(a) Draw a sketch of $\mathbf{F}$ on the region where $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. Check you answer with the instructor.

(b) Consider the following two curves which start at $A = (-2, 0)$ and end at $B = (2, 0)$, namely the line segment $C_1$ and upper semicircle $C_2$.

Add these curves to your sketch, and compute both $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Check you answers with the instructor.

(c) Based on your answer in (b), could $\mathbf{F}$ be $\nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$? Explain why or why not.

Note: More problems on the back.
3. Consider the points $A = (0, 0)$ and $B = (\pi, -2)$. Suppose an object of mass $m$ moves from $A$ to $B$ and experiences the constant force $\mathbf{F} = -mg\mathbf{j}$, where $g$ is the gravitational constant.

(a) If the object follows the straight line from $A$ to $B$, calculate the work $W$ done by gravity using the formula from the first week of class.

(b) Now suppose the object follows half of an inverted cycloid $C$ as shown below. Explicitly parameterize $C$ and use that to calculate the work done via a line integral.

(c) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ so that $\nabla f = \mathbf{F}$. Use the Fundamental Theorem of Line Integrals to check your answers for (a) and (b). Have you seen the quantity $-f$ anywhere before? If so, what was its name?

4. If you get this far, work #48 from Section 16.2:

48. Experiments show that a steady current $I$ in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). **Ampère’s Law** relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where $I$ is the net current that passes through any surface bounded by a closed curve $C$, and $\mu_0$ is a constant called the permeability of free space. By taking $C$ to be a circle with radius $r$, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance $r$ from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$