Discussion — Thursday, October 7th

Subject: Curves and integration.

1. (a) Sketch the first-octant portion of the sphere \( x^2 + y^2 + z^2 = 16 \). Check that \( P = (2, 2, 2\sqrt{2}) \) is on this sphere and add this point to your picture.

(b) Find a function \( f(x, y) \) whose graph is the top-half of the sphere.

(c) Imagine an ant walking along the surface of the sphere. It walks down the sphere along the path \( C \) that passes through the point \( P \) in the direction parallel to the \( yz \)-plane. Draw this path in your picture.

(d) Use the function from (b) to find a parameterization \( r \) of the ant's path along the portion of the sphere shown in your picture. Specify the domain for \( r \), i.e. the initial time when the ant is at \( P \) and the final time when it hits the \( xy \)-plane.

(e) Adjust your parameterization so that the ant is at \( P \) at time 0 and hits the \( xy \)-plane at time 1. Hint: See 2(b) below. Check your answer with the instructor.

2. Consider the curve \( C \) in \( \mathbb{R}^3 \) given by
   \[
   \mathbf{r}(t) = (e^t \cos t) \mathbf{i} + 2\mathbf{j} + (e^t \sin t) \mathbf{k}
   \]

(a) Calculate the length of the segment of \( C \) between \( \mathbf{r}(0) \) and \( \mathbf{r}(t_0) \). Check your answer with the instructor.

(b) Suppose \( h: \mathbb{R} \to \mathbb{R} \) is a function. We can get another parameterization of \( C \) by considering the composition
   \[
   f(s) = r(h(s))
   \]
   This is called a reparameterization. Find a choice of \( h \) so that
   i. \( f(0) = \mathbf{r}(0) \)
   ii. The length of the segment of \( C \) between \( f(0) \) and \( f(s) \) is \( s \). (This is called parameterizing by arc length.)

   Check your answer with the instructor.

(c) Without calculating anything, what is \( |f'(s)| \)?

(d) Draw a sketch of \( C \).

3. Consider the curve \( C \) given by the parameterization \( \mathbf{r}: \mathbb{R} \to \mathbb{R}^3 \) where \( \mathbf{r}(t) = (\sin t, \cos t, \sin^2 t) \).

(a) Show that \( C \) is in the intersection of the surfaces \( z = x^2 \) and \( x^2 + y^2 = 1 \).

(b) Use (a) to help you sketch the curve \( C \).

4. As in 2(b), consider a reparameterization
   \[
   f(s) = r(h(s))
   \]
   of an arbitrary vector-valued function \( \mathbf{r}: \mathbb{R} \to \mathbb{R}^3 \). Use the chain rule to calculate \( |f'(s)| \) in terms of \( \mathbf{r}' \) and \( h' \).