Lecture 6: Level sets in 3-dimensional space (14.1),
quadric surfaces (12.6), review of limits (14.2)

Last time: \( f: \mathbb{R}^2 \to \mathbb{R} \quad f(x,y) = x^2 - y^2 \)

Graph

Level sets

For \( f: \mathbb{R}^3 \to \mathbb{R} \) we can't draw the graph
(in \( \mathbb{R}^4 \)) but can still look at level sets.

\[ \text{[Did } f(x,y,z) = x^2 + y^2 + z^2 \text{ last time]} \]

\( \text{Ex: } f(x,y,z) = x^2 + y^2 - z^2 \)

First, look at \( xz \)-plane

\( f(x,0,z) = x^2 - z^2 \)

so the level sets
in this plane look like what we had on Wed.

Also, as \( x^2 + y^2 = r^2 \)

we have \( f(x, y, z) = r^2 - z^2 \) and so each

level set is symmetric about \( z \)-axis:

\[
\begin{align*}
\text{f} &= 1 \\
\text{f} &= 0 \\
\text{f} &= -1
\end{align*}
\]

[These level sets are all examples of quadric surfaces.]
Conic Sections: Solutions in $\mathbb{R}^2$

of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Circle $\rightarrow$ Ellipse $\rightarrow$ Parabola $\rightarrow$ Hyperbola

Quadric Surfaces in $\mathbb{R}^3$ (Section 12.6)

$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$

Ex: Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Elliptic paraboloid:

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Hyperbolic paraboloid:

$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
The other quadric surfaces are the double cone and the hyperboloid (of 1 and 2 sheets) that we saw at the start.

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Limits (14.2) [To talk about derivatives, first need to discuss limits for functions of several variables]

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

[Can take different perspectives on limits — focus on them as a way of estimating errors.]

What does a calculator do to compute

\[ \sin(2) = 0.9092974268 \]

Uses

\[ \sin(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]

But how many terms do we need to add up?
Consider $E : \mathbb{R} \to \mathbb{R}$ (an "error function")

We say

$$\lim_{h \to 0} E(h) = 0$$

if given $\exists > 0$ we can always find $S > 0$ so that whenever $|h| < S$ then $|E(h)| < \varepsilon$.

$E(h) = h^2$

View as challenge-response process.

**Ex:** $E(h) = h^2$  $\varepsilon = 1/10$

Take $S = 1/4$. If $|h| < S = 1/4$, then

$$|E(h)| = |h^2| = |h|^2 < \frac{1}{16} < \frac{1}{10}$$

**Ex:** $\varepsilon = 1/100$  $S =$ [Audience response]

$\varepsilon = 1/1000$  $S =$ [— — — — — —]
Claim: \( \lim_{h \to 0} h^2 = 0 \)

Reason: If you give me \( \varepsilon > 0 \), we'll take \( \delta = \sqrt{\varepsilon} \). Then if \( |h| < \delta \) we have

\[
|h^2| = |h|^2 < \delta^2 = \varepsilon.
\]

**Ex:** \( E(h) = 2h + h^2 \)  
Know \( \lim_{h \to 0} 2h + h^2 = 0 \)

Given \( \varepsilon = \frac{1}{10} \) take \( \delta = \frac{1}{100} \).

If \( |h| < \delta \), then \( |2h + h^2| \leq 2|h| + |h|^2 \) < \( \frac{3}{100} < \frac{1}{10} = \varepsilon \).

In general, say

\[
\lim_{x \to a} f(x) = C
\]

if \( f(a + h) = C + E(h) \)

where \( \lim_{h \to 0} E(h) = 0 \).

Differentiability:

\[
f(x+h) = f(x) + f'(x)h + E(h)
\]

where \( E(h) \) is really small, i.e.

\[
\lim_{h \to 0} \frac{E(h)}{h} = 0.
\]