Lecture 11: The chain rule (14.5) and directional derivatives (14.6).

Last time: Chain rule

1. \( f: \mathbb{R}^2 \to \mathbb{R}, \ x, y: \mathbb{R} \to \mathbb{R} \)
   Consider \( h(t) = f(x(t), y(t)) \). Then
   \[
   h'(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)
   \]

2. \( z = f(x, y) \) with \( x = x(t) \) and \( y = y(t) \). Then
   \[
   \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
   \]

What if \( x \) and \( y \) themselves depend on more than one variable?

Ex: \( z = f(x, y) = x^2 + 3y \)
   \( x(r, \theta) = r \cos \theta \)
   \( y(r, \theta) = r \sin \theta \)

So \( z(r, \theta) = f(x(r, \theta), y(r, \theta)) = r^2 \cos^2 \theta + 3r \sin \theta \)

Chain Rule: \[
\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}
\]
\[= 2x \cdot (-r \sin \theta) + 3 \cdot (r \cos \theta)\]

\[= -2r^2 \sin \theta \cos \theta + 3r \cos \theta\]

[Easy to see this matches what we would get by doing it directly.]

Ex: \(U = f(x, y, z)\)
\[x = x(s, t)\]
\[y = y(s, t)\]
\[z = z(s, t)\]

Chain Rule: \(\frac{du}{ds} = \frac{du}{dx} \frac{dx}{ds} + \frac{du}{dy} \frac{dy}{ds} + \frac{du}{dz} \frac{dz}{ds}\)

Directional Derivatives: (14.6) \(f: \mathbb{R}^2 \rightarrow \mathbb{R}\)

[Already have \(\partial\) derivatives, measuring change in (\(x,y\)) directions. But why give axes special treatment?]

Pick a point \(\vec{a}\) in \(\mathbb{R}^2\) and a direction \(\vec{v}\).
The derivative of $f$ in direction $\vec{v}$ at $\vec{a}$ is the rate of change in $f$ as we move in direction $\vec{v}$ away from $\vec{a}$.

$$D_{\vec{v}} f(\vec{a}) = \lim_{t \to 0} \frac{d}{dt} f(\vec{a} + t \vec{v}) \bigg|_{t=0}$$

fn of var

Ex: If $\vec{v} = \vec{1}$, then $D_{\vec{1}} f(\vec{a}) = \frac{\partial f}{\partial x}(\vec{a})$

In general can find via the chain rule:

$$\vec{a} = (a_1, a_2) \quad \vec{v} = (v_1, v_2) \quad \vec{a} + t \vec{v} = (a_1 + tv_1, a_2 + tv_2)$$

$$f(\vec{a} + t \vec{v}) = f(x, y) \quad \text{where} \quad x = a_1 + tv_1, \quad y = a_2 + tv_2$$
Now: \( f'(0) = \frac{\partial f}{\partial x}(x(0), y(0)) \cdot x'(0) + \frac{\partial f}{\partial y}(x(0), y(0)) \cdot y'(0) \)

\[ = \frac{\partial f}{\partial x}(a_1, a_2) V_1 + \frac{\partial f}{\partial y}(a_1, a_2) V_2 \]

\[ = D_v f(\bar{a}). \]

\text{Ex:} \quad f(x, y) = x^2 + y^2 \quad \begin{aligned} \bar{u} &= \frac{1}{\sqrt{2}} \hat{e} + \frac{1}{\sqrt{2}} \hat{j} \end{aligned} \]

\[ D_{\bar{u}} f(2, 1) \]

\[ = \frac{\partial f}{\partial x}(2, 1) \cdot \frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(2, 1) \cdot \frac{1}{\sqrt{2}} \]

\[ = 4 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}(3) = 3\sqrt{2}. \]
Gradient: $f: \mathbb{R}^2 \to \mathbb{R}$

$$\nabla f(\bar{a}) = \left( \frac{\partial f}{\partial x}(\bar{a}), \frac{\partial f}{\partial y}(\bar{a}) \right)$$

[Will give a geom. interp. in a minute, but for now:]

$$D_{\vec{v}} f(\bar{a}) = \nabla f(\bar{a}) \cdot \vec{v}$$

Suppose we want to know: in which direction does $f$ increase fastest?

Suppose $\vec{v}$ is a unit vector.

Then

$$D_{\vec{v}} f(\bar{a}) = \nabla f(\bar{a}) \cdot \vec{v} = |\nabla f(\bar{a})| \cos \theta$$

So to maximize, we want $\theta = 0$, i.e. $\vec{v}$ points in the same dir. as $\nabla f$, and so

\[\overrightarrow{\text{Max}}\]
Thus: $\nabla f(\hat{a})$ points in direction of fastest increase of $f$. Length is rate of said increase.

Ex: $f(x,y) = 1 - 4x^2 - y^2$

$\nabla f = (-8x, -2y)$

Level sets:

$f = 0 \quad 4x^2 + y^2 = 1$

$f = -3 \quad 4x^2 + y^2 = 4$

What is $\nabla f(\hat{a})$?

A: $\hat{0}$.

Moral: A min/max can only occur when $\nabla f = \hat{0}$.