Lecture 7: Continuity (14.2) and partial derivatives (14.3).

Show computer plots of \( \frac{2xy}{x^2+y^2} \) and \( \frac{xy^2}{x^2+y^4} \)

**Continuity:** \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is continuous at \( \tilde{a} \) if \( \lim_{\tilde{x} \rightarrow \tilde{a}} f(\tilde{x}) = f(\tilde{a}) \)

Some meanings:

1. Can evaluate limits by plugging in: \( f(x,y) = \frac{x+1}{y} \)

\[
\lim_{(x,y) \rightarrow (1,2)} f(x,y) = 1 = f(1,2) = \frac{1+1^2}{2} = 1.
\]

From last time, using limit laws

\[ f(\tilde{a} + \tilde{h}) = f(\tilde{a}) + E(\tilde{h}) \text{ where } \lim_{\tilde{h} \rightarrow \delta} E(\tilde{h}) = 0. \]
Most, but not all, functions you encounter in nature are continuous. For example:
functions built from other continuous functions via $+,-,\times,\div$ (but not by $0$), composition (e.g. $\sqrt{x^2+y^2}$)

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

At $(x,y) \neq (0,0)$ the fn $f$ is cont as it is built up from continuous pieces.

At $(0,0)$ need to check directly that

$$\lim_{(x,y) \to (0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0 = f(0,0)$$

Reason: Suppose $\varepsilon > 0$. Take $\delta = \varepsilon$.
If $\bar{h} = (x,y)$ sat $0 < |\bar{h}| < \delta$, then as $|x| \leq |\bar{h}|$

$$\left| \frac{x^2}{\sqrt{x^2+y^2}} \right| = \frac{|x|^2}{|\bar{h}|} \leq |\bar{h}| < \delta = \varepsilon.$$
**Derivatives:** For \( f: \mathbb{R} \to \mathbb{R} \) this is about approximation by lines. Can't always do:

\[
\begin{align*}
\text{Or even do anywhere:} \\
\end{align*}
\]

**Examples:** Stock market; Brownian motion.

Think dust moving in sunlight. Brown (19th cent.) observed with pollen moving on the surface of water. Einstein (1905) brought to attention of physicists. 2000 years earlier, the Roman Lucretius used this idea to...
argue for the existence of molecules...

In this class we will work almost exclusively with functions that have derivatives.

**Partial derivatives:** \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \)

At what rate does \( f \) change if we start at \((a, b)\) and vary the \( x \)-coordinate?

\[
\frac{\partial f}{\partial x} (a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}
\]

Compare

\[
f'(a) = \frac{df}{dx} (a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]
Easy to compute: Just view \( y \) as a constant and differentiate with respect to \( x \).

\[
\frac{\partial f}{\partial x} = 2x + y + 0 \quad \frac{\partial f}{\partial x}(3,1) = 7
\]

Can also look at the rate of change in the \( y \)-direction: [View \( x \) as a constant]

\[
\frac{\partial}{\partial y} \left( (x+y) \sin(xy) \right)
\]

\[
= \left( \frac{\partial}{\partial y} (x+y) \right) \sin(xy) + (x+y) \frac{\partial}{\partial y} (\sin(xy))
\]

\[
= \sin(xy) + (x+y) \cos(xy) x
\]
Other notation:

\[ \frac{df}{dx}(a,b) = \frac{\partial}{\partial x} f(a,b) = f_x(a,b) = D_1 f(a,b) \]

Partial Differential Equations:

O.D.E. ① \( P(t) = \text{pop at time } t \)

\[ P'(t) = c \cdot P(t) \implies P(t) = P_0 e^{ct} \]

\[ h''(t) = -g - \alpha \cdot h'(t) \]
\[ h(t) = -\frac{1}{\alpha^2} \left( at + \left( av_0 + g \right) e^{-at} \right) \]

\[ F = ma \]

P.D.E.

\[ u(x,t) = \text{temp of rod at pos } x \text{ and time.} \]
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} u \right) \]

Heat eqn

Comes from Newton's law of cooling: Flow
heat is prop.
to \(-\frac{\partial u}{\partial x}\)

\[ \frac{\partial^2 u}{\partial x^2} < 0 \]

\( u(x, t_0) \)

\( \frac{\partial^2 u}{\partial x^2} > 0 \)

\[ u \]

\[ u(t) \]

\[ \partial \]

\[ = \]

\[ t \]
Skip to here if P.D.E. discussion is going to be too long:

Next time, tangent plane.

\[ \frac{df}{dx}(a,b) = \text{slope in the slice } y = b \]