Lecture 18: Integration along curves (13.3 and 16.2)

Last time: \( \vec{r} : \mathbb{R} \to \mathbb{R}^2 \) or \( \mathbb{R}^3 \)

Length: \( \vec{r} : [a, b] \to \mathbb{R}^3 \)

\[
\text{Length} = \int_a^b |\vec{r}'(t)| \, dt.
\]

Ex: Cycloid \( \vec{r}(t) = (t - \sin t, 1 - \cos t) \)

\[
\vec{r}'(t) = (1 - \cos t, \sin t)
\]

\[
|\vec{r}'(t)| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2 \cos t + \cos^2 t}
\]

\[
= \sqrt{2 - 2 \cos t} = \sqrt{4 \sin^2 t/2} = 2 \sin t/2
\]

Since

\[
\cos 2s = \cos^2 s - \sin^2 s
\]

\[
= 1 - 2 \sin^2 s
\]

\[
\Rightarrow 2 \sin^2 s = 1 - \cos 2s
\]

Thus

\[
\text{Length} = \int_0^{2\pi} |\vec{r}'(t)| \, dt = \int_0^{2\pi} 2 \sin t/2 \, dt = -4 \cos t/2 \bigg|_{t=0}^{2\pi} = 4 - (-4) = 8.
\]
Really two closely related concepts:

Curve: A set of points in $\mathbb{R}^3$ looking like:

Parameterization:

$\mathbf{r}: \mathbb{R} \to \mathbb{R}^3$

Instructions for moving along a curve

[Any curve has many parameterizations]

Ex:

$C = \frac{1}{4}$ of unit circle in $\mathbb{R}^2$

(a) $\mathbf{r}: [0, \pi/2] \to \mathbb{R}^2$

$\mathbf{r}(t) = (\cos t, \sin t)$

$\mathbf{r}'(t) = (-\sin t, \cos t)$

Length $= \int_0^{\pi/2} \sqrt{(-\sin t)^2 + \cos^2 t} \, dt = \frac{\pi}{2}$

(b) $\mathbf{r}: [0, 1] \to \mathbb{R}^2$

$\mathbf{r}(t) = (t, \sqrt{1-t^2})$

$\mathbf{r}'(t) = (1, \frac{-t}{\sqrt{1-t^2}})$

Length $= \int_0^1 \sqrt{1 + \frac{t^2}{1-t^2}} \, dt$

$= \arcsin (t) \bigg|_0^{\pi/2} = \frac{\pi}{2}$
Notice that the length only depended on the curve, not the parameterization.

Another point of view: Approximate by straight segments.

Linear approximation:

\[ \vec{r}(t_i + \Delta t) \approx \vec{r}(t_i) + \vec{r}'(t_i) \Delta t \]

Take \( \Delta t = t_{i+1} - t_i \) so get

\[ \vec{r}(t_{i+1}) - \vec{r}(t_i) \approx \vec{r}'(t_i) \Delta t \]. Thus

\[
\text{Length of } C \approx \sum_{i=0}^{n} |\vec{r}(t_{i+1}) - \vec{r}(t_i)| \approx \sum_{i=0}^{n} \vec{r}'(t_i) \Delta t
\]
As $\Delta t \to 0$, the Riemann sums at right converge to $\int_a^b |F'(t)| \, dt$. As the other approx get better as $\Delta t$, we've found again that

$$\text{Length of } C = \int_a^b |F'(t)| \, dt$$

Integration along a curve: (Line integral 16.2)

$C$ curve in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$ a function

$$\int_C f \, ds = \int_a^b f(r(t)) |r'(t)| \, dt$$

where $\vec{r}: [a, b] \to \mathbb{R}^2$ is a parameterization of $C$.

[Turns out not to depend on $\vec{r}$, just $C$]
Some meanings:

1. $f =$ temperature

$$\text{Average temp} = \frac{1}{\text{Length } C} \int_C f \, ds$$

2. $f =$ density of curve (mass/unit length)

$$\text{mass of curve} = \int_C f \, ds$$

3. Area of region above $C$ and below the graph of $f$. 

Diagram of a region in the $xy$-plane with a curve $C$. 

Value: 57.
Ex: Find the average of \( f(x,y) = 10x^2 \) on \( C \)

\[ \vec{r}(t) = (\cos t, \sin t) \quad 0 \leq t \leq \frac{\pi}{2} \]

\[
\int_C f \, ds = \int_0^{\pi/2} f(\vec{r}(t)) |\vec{r}'(t)| \, dt
\]

\[
= \int_0^{\pi/2} 10\cos^2(t) \cdot 1 \, dt = 10 \int_0^{\pi/2} \cos^2 t \, dt
\]

\[
= \frac{5}{2} \pi
\]

Note: \( \cos^2 t = \frac{1}{2} (1 + \cos 2t) \), so

\[
\int_0^{\pi/2} \cos^2 t \, dt = \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2t) \, dt
\]

\[
= \frac{1}{2} \left( t + \frac{1}{2} \sin 2t \right) \bigg|_{t=0}^{t=\pi/2} = \frac{\pi}{4}
\]

Average = \( \frac{1}{\text{Length}} \int_C f \, ds = \frac{\frac{5}{2} \pi}{\frac{\pi}{2}} = 5 \)