Lecture 23: Conservative vector fields II (16.3) 70

Last time: \( \vec{F}: \mathbb{R}^n \to \mathbb{R}^n \) is conservative if \( \vec{F} = \nabla f \) for some \( f: \mathbb{R}^n \to \mathbb{R} \).

Note: HW for today included w/ last assig.

Thm A: \( \vec{F} \) a vector field on an open connected region \( D \) in \( \mathbb{R}^2 \).
\( \vec{F} \) is conservative if and only if \( \int_C \vec{F} \cdot d\vec{r} \) is path independent.

Thm B: \( \vec{F} = (P, Q) \) on simply connected open \( D \) in \( \mathbb{R}^2 \).
\( \vec{F} \) is conserv. if and only if \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \) on \( D \).

Thm A also works for \( \mathbb{R}^n \).

Thm B has analogs, but more complicated. (16.3 #27)

[Note to self: right cond is \( H'(D; \mathbb{R}) = 0 \).]

Do physics first!

Reason for Thm A: Suppose \( \int_C \vec{F} \cdot d\vec{r} \) is path-indep.

Pick \( \vec{x}_0 \) in \( D \)

Define

\( f: D \to \mathbb{R} \)

by

\[ C \quad \vec{x} \quad \vec{x}_0 \]
\[ f(\vec{x}) = \int_C \vec{F} \cdot d\vec{r} \quad \text{for any path } C \text{ from } \vec{x}_0 \text{ to } \vec{x}. \]

Note: \( f(\vec{x}_0) = 0 \) and so the F.T.I.L.I says:

\[ \int_C \nabla f \cdot d\vec{r} = f(\vec{x}) - f(\vec{x}_0) = f(\vec{x}) \]

Point: \( \nabla f = \vec{F} \). For instance, let's compute

\[ \frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{C_2} \vec{F} \cdot d\vec{r} \]

\[ f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r} \quad \text{at } (x,y) \]

\[ f(x+h,y) = \int_{C_1+C_2} \vec{F} \cdot d\vec{r} \quad \text{at } (x+h,y) \]
Now if \( \vec{F} = (P, Q) \) then as \( \vec{T} = (1, 0) \)

\[
\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot \vec{T} \, ds = \int_{C_2} P \, ds
\]

So

\[
\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{1}{h} \int_{C_2} P \, ds = P(x, y).
\]

Averaging over shorter and shorter curves.

Likewise \( \frac{\partial f}{\partial y} = Q \) and so \( \nabla f = \vec{F} \), i.e. \( \vec{F} \) is conservative.

Physical motivation: Force \( \vec{F} = -\frac{MmG}{r^2} \), \( r \) is the distance.

\[
V = -\frac{MmG}{r}
\]

Potential Function

Note: \( \vec{F} = -\nabla V \)

\( V \) is larger the farther away the smaller mass is.

\( \vec{F} \) points in the direction of fastest decrease in \( V \).
More generally, suppose an object's path is $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ acted on by a conservative force, i.e.

$$\vec{F} = -\nabla V$$

for some $V : \mathbb{R}^2 \to \mathbb{R}$.

Newton's Law: $\vec{F} = m\vec{a}$ or

$$\vec{F}(\vec{r}(t)) = m\vec{r}''(t)$$

**Total Energy:**

$$E(t) = (\text{Kinetic energy}) + (\text{potential energy})$$

$$= \frac{1}{2} m |\vec{r}'(t)|^2 + V(\vec{r}(t))$$

**Conservation of Energy:** $E(t)$ is const. indep. of $t$.

**Reason:** Let's compute $E'(t)$. If $\vec{F} = (r_1, r_2)$

$$\frac{d}{dt} \left( \frac{1}{2} m |\vec{r}'(t)|^2 \right) = \frac{1}{2} m \cdot \frac{d}{dt} \left( (r_1'(t))^2 + (r_2'(t))^2 \right)$$

$$= \frac{1}{2} m \left( 2r_1'(t)r_1''(t) + 2r_2'(t)r_2''(t) \right)$$
\[ \vec{F}''(t) \cdot \vec{F}'(t) \]

and

\[ \frac{d}{dt} V(\vec{F}(t)) = \text{rate V changes as a fn of } t \]

\[ = \nabla V(\vec{F}(t)) \cdot \vec{F}'(t) \]

So

\[ E'(t) = m \vec{F}''(t) \cdot \vec{F}'(t) + \nabla V(\vec{F}(t)) \cdot \vec{F}'(t) \]

\[ \uparrow \text{Newton!} \]

\[ = \vec{F}(\vec{F}(t)) \cdot \vec{F}'(t) - \vec{F}(\vec{F}(t)) \cdot \vec{F}'(t) \]

\[ = 0. \]

Hence energy is conserved.