Lecture 22: Conservative Vector Fields (16.3)

Previously on Math 241:

Fund. Thm. of Line Integrals: \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) differentiable

\[ \int_C \nabla f \cdot dr = f(B) - f(A) \]

A vector field \( \vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is conservative if \( \vec{F} = \nabla f \)

for some \( f: \mathbb{R}^n \rightarrow \mathbb{R} \)

By the Fund. Thm, if \( \vec{F} \) is conservative then

(a) Independence of path: If \( C_1 \) and \( C_2 \)

are two paths joining \( A \) to \( B \), then

\[ \int_{C_1} \vec{F} \cdot dr = \int_{C_2} \vec{F} \cdot dr \]

(b) If \( C \) is a closed curve (i.e. starts and ends at the same pt, like a circle) then

\[ \int_C \vec{F} \cdot dr = 0 \]
Reason: If \( \vec{F}(a) = \vec{F}(b) \), then

\[
\int_C \nabla f \cdot d\vec{r} = f(\vec{F}(b)) - f(\vec{F}(a)) = 0
\]

(Note: Actually, these two conditions are equivalent. For example, if (b) is true for \( \vec{F} \), consider)

\[\begin{array}{c}
\text{A} \\
C_1 \\
C_2
\end{array}\]

\[\begin{array}{c}
\text{B} \\
\text{Let } -C_2 \text{ denote } C_2 \text{ backwards}
\end{array}\]

\[\begin{array}{c}
\text{Take } C = C_1 \\
-C_2
\end{array}\]

Then

\[
0 = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}
\]

\[
= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}
\]

\[
\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}
\]

(Conversely, if (a) holds for \( \vec{F} \), then break a closed curve into 2 segments to see that \( \int_C \vec{F} \cdot ds = 0 \).)
Examples of nonconservative vector fields:

1. \( \vec{F} = (y, 0) \) [from yesterday's worksheet]
   \[
   \int_{C_1} \vec{F} \cdot d\vec{r} = 0 \quad \text{different!}
   \]
   \[
   \int_{C_2} \vec{F} \cdot d\vec{r} = 2\pi
   \]

2. \( \vec{F} = (-y, x) \) [from 1st lecture on vector fields.]
   \[
   \vec{F} = \vec{\nabla} f \quad \int_{C} \vec{F} \cdot d\vec{r} = \int_{C} \vec{\nabla} f \cdot d\vec{r} = \int_{C} df = \int_{C} ds = 2\pi \neq 0
   \]

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Note: Being conservative is kinda subtle. For example, is \( \vec{F} = (y, x) \) conservative? [Compare 2 above]

A. Yes. Need \( f(x, y) \) with \( \frac{\partial f}{\partial x} = y \) and \( \frac{\partial f}{\partial y} = x \)

First cond says \( f = \int y \, dx = xy + C(y) \)

\( \uparrow \) treated as a constant

Second is similar, so take \( f = xy \).
Usefulness is clear:

\[ \int_C \vec{F} \cdot ds = f(3,1) - f(0,0) = 3. \]

[Will give two tests for \( \vec{F} \) to be conservative, but]
[First need to introduce some terms...]

Focus on a vector field \( \vec{F} \) on \( \mathbb{R}^2 \), with domain \( D \).

Ex: \( \vec{F} = (-y, x) \)  \( D = \mathbb{R}^2 \)

\[ \vec{F} = \frac{1}{x^2 + y^2} (-y, x) \quad D = \{(x, y) \neq (0,0)\} \]

Properties \( D \) may have:

Open: Roughly, \( D \) contains none of its boundary points. Precisely, each pt of \( D \) is the center of a disc also contained in \( D \).

Ex: \( D = \{(x, y) \mid 0 < x < 1 \} \)

\( D = \mathbb{R}^2 \)
\( D = \{(x, y) \neq (0,0)\} \)
Closed: Opposite of open.

[D is closed if and only if its complement is open.]

Connected: Any two points in D can be joined by a path inside D.

Simply Connected: Connected + no holes.

Ex:  
Non Ex:  

\[ \{(x, y) \neq (0, 0)\} \]

ThmA: Suppose \( \vec{F} \) is a vector field on an open connected set D. Then \( \vec{F} \) is conservative if and only if \( \int_C \vec{F} \cdot d\vec{r} \) is path independent.

ThmB: Suppose \( \vec{F} = (P, Q) \) is a vector field on an open simply connected set D. Then \( \vec{F} \) is conservative if and only if \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \) on D.
Reason for Thm B: Suppose \( \vec{F} = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \)

Then

\[
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}
\]

and

\[
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}
\]

and mixed partials are equal!

[We'll come back to why \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} \) conserv.
just before Thanksgiving...]

Ex: \( \vec{F} = (y, x) \) and \( \frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} \Rightarrow \) conserv.

Ex: \( \vec{F} = (-y, x) \) and \( \frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial Q}{\partial x} \Rightarrow \) not conserv.

Ex: \( \vec{F} = \frac{1}{x^2+y^2}(-y, x) \) on \( D = \{ (x,y) \neq 0 \} \)

On HW: Check that \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \) but not path indep, hence not conservative.

Point: \( D \) is not simply conn.