Lecture 30: Changing coordinates I (section 15.9)

Previously: \[ \Theta \quad \frac{2\pi}{2\pi} \quad T \]

\[ T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad T(r, \Theta) = (r \cos \Theta, r \sin \Theta) \]

\[ \iint_{R} f(x, y) \, dA = \iint_{S} f(T(r, \Theta)) \, rdrd\Theta \]

Next two lectures: general change of coordinates...

Suppose we want to integrate many functions over the region shown at right. For each, would need two integrals to do so:

\[ \iint_{R} f(x, y) \, dA = \int_{0}^{2} \int_{\frac{x}{2}}^{x} f(x, y) \, dy \, dx + \int_{2}^{3} \int_{2x-3}^{x} f(x, y) \, dy \, dx \]

Goal: Do a change of coordinates so that can use just one integral. Sing praises thereof.
Need \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) so that \( T(S) = \mathbb{R} \).

Simplist kind of \( T \): Linear Transformations.

\[
T_A(u, v) = (au + bv, cu + dv)
\]

for some \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

Ex: \( T(u, v) = (u - 2v, u + 2v) \)

from Thursday worksheet.

Key properties:

a) \( T(\text{Line}) = \text{Line} \)

b) \( T(0,0) = (0,0) \)

c) \( T \) det. by \( T(1,0) = (a,c) \) and \( T(0,1) = (b,d) \)

d) \( \overrightarrow{w_1}, \overrightarrow{w_2} \) in \( \mathbb{R}^2 \), \( s, t \) in \( \mathbb{R} \)

\[
T(s\overrightarrow{w_1} + t\overrightarrow{w_2}) = sT(\overrightarrow{w_1}) + tT(\overrightarrow{w_2})
\]
If we want $T(S) = R$

then $T(1,0) = (2,1)$ and $T(0,1) = (3,3)$.

Hence $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $T(u,v) = (2u+3v, u+3v)$

if $T$ is linear, and that will work since such send lines to lines.

To integrate, need to understand how $T$ distorts area.

Now a linear transform distorts area uniformly

(think about problem 1 on the worksheet)
Hence \[ \text{Area} \left( \square \right) = \Delta u \Delta v \quad \rightarrow \quad \text{Area} \left( \triangle \right) = 3 \Delta u \Delta v \]

So: \[ dA = 3 \, du \, dv \]

Ex: \[ \iint_R x - y \, dA = \iint_S (2u+3v)-(u+3v) \, 3 \, du \, dv \]

\[ T(u,v) = (2u+3v, u+3v) = (x, y) \]

\[ = \int_0^1 \int_0^{1-u} u \cdot 3 \, dv \, du = \int_0^1 3u(1-u) \, du \]

\[ = \int_0^1 3u - 3u^2 \, du = \frac{3}{2} u^2 - u^3 \bigg|_{u=0}^{u=1} = \frac{1}{2} . \]

Fun check: Do as shown on pg 89.
In general, if \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is linear, with \( T(u, v) = (a \, u + b \, v, c \, u + d \, v) \) for \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then
\[
\begin{align*}
\int_{S} f(x, y) \, dA &= \int_{S} f(T(u, v)) \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, du \, dv \\
\end{align*}
\]
and \( dA = |a \, b| \, du \, dv \).

Thus,
\[
\int_{R} f(x, y) \, dA = \int_{S} f(T(u, v)) \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, du \, dv
\]

Linear Approx: Recall if \( g : \mathbb{R}^2 \to \mathbb{R} \) is diff at \((u, v)\) then
\[
g(u + \Delta u, v + \Delta v) = g(u, v) + g_u(u, v) \Delta u + g_v(u, v) \Delta v + E(\Delta u, \Delta v)
\]
small.

Now consider \( T : \mathbb{R}^2 \to \mathbb{R}^2 \)
\[
T(u, v) = (g(u, v), \ h(u, v))
\]
Then

\[ T(u + \Delta u, v + \Delta v) = T(u, v) + J_{(u,v)}(\Delta u, \Delta v) + \text{Error} \]

where

\[ J_{(u,v)} \text{ is the linear transformation with matrix } \begin{pmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{pmatrix} \]

Reason

\[ T(u + \Delta u, v + \Delta v) \]

\[ \approx (g(u,v) + g_u(u,v)\Delta u + g_v(u,v)\Delta v, \\
    h(u,v) + h_u(u,v)\Delta u + h_v(u,v)\Delta v) \]

Thus locally \( T \) looks like a linear transformation...

--- to be continued ---