Last time: Stokes' Thm: \( S \) a surface in \( \mathbb{R}^3 \)
\( \mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3 \) a vector field. Then
\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl} \, \mathbf{F}) \cdot \hat{n} \, dA
\]

Ex: \( \mathbf{F} = (y, xz, 1) \quad \text{curl} \, \mathbf{F} = (-x, 0, z-1) \)

\[
\iint_S (\text{curl} \, \mathbf{F}) \cdot \hat{n} \, dA = -\pi \quad \text{for all of these!}
\]

\[
= \oint_C \mathbf{F} \cdot d\mathbf{r}
\]

Easy to check for \( D \):
\[
\iint_D (\text{curl} \, \mathbf{F}) \cdot \hat{n} \, dA = \iint_D (-x, 0, -1) \cdot (0,0,1) \, dA =
\]
\[
= \iint_D -1 \, dA = -\text{Area}(D) = -\pi \checkmark
\]
Note: Stokes' Thm also works when $S$ has several boundary components [provided they are oriented correctly.]

Understanding Curl:
\[ \mathbf{F} = \text{velocity of fluid flow} \]

Consider a small paddle wheel:

\[ \overrightarrow{n} = \text{unit vector} \]
\[ \mathcal{D}_r = \text{disc of radius } r, \quad \perp \text{ to } \overrightarrow{n} \]

Key: Wheel rotates at
\[ \omega = \frac{1}{2\pi r^2} \int_{\mathcal{C}_r} \mathbf{F} \cdot d\mathbf{r} \]

Reason: Suppose \( \overrightarrow{n} = (0,0,1) \) and the wheel is rotating at a constant rate \( \omega \):
\[ \mathcal{C}(t) = (r \cos \omega t, r \sin \omega t, 0) \]

If the tangential component of \( \mathbf{F} \) has the same length everywhere,
\[ \mathbf{\tilde{z}}'(t) = \text{Proj}_{\mathbf{\tilde{z}}(t)} \mathbf{\tilde{F}} = \frac{\mathbf{\tilde{F}} \cdot \mathbf{\tilde{z}}'(t)}{|\mathbf{\tilde{z}}'(t)|^2} \]

i.e. \( \mathbf{\tilde{F}} \cdot \mathbf{\tilde{z}}'(t) = |\mathbf{\tilde{z}}'(t)|^2 \). In general the average of these should be the same, so

\[
\int_0^{2\pi/\omega} \mathbf{\tilde{F}} \cdot \mathbf{\tilde{z}}'(t) \, dt = \int_0^{2\pi/\omega} |\mathbf{\tilde{z}}'(t)|^2 \, dt = \int_0^{2\pi/\omega} r^2 \omega^2 \, dt
\]

\[
= 2\pi r^2 \omega
\]

\[
\int_{C_r} \mathbf{\tilde{F}} \cdot d\mathbf{\tilde{r}} \]

Thus \( \omega = \frac{1}{2\pi r^2} \int_{C_r} \mathbf{\tilde{F}} \cdot d\mathbf{\tilde{r}} \)

By Stokes:

\[
\omega = \frac{1}{2\pi r^2} \iint_{D_r} (\text{curl} \mathbf{\tilde{F}}) \cdot \mathbf{\hat{n}} \, dA
\]

\[
= \frac{1}{2} \left( \frac{1}{\text{Area}(D_r)} \iint_{D_r} (\text{curl} \mathbf{\tilde{F}}) \cdot \mathbf{n} \, dA \right)
\]

Taking \( r \to 0 \), get

\[
\omega = \frac{1}{2} \left( \text{curl} \mathbf{\tilde{F}}(p) \right) \cdot \mathbf{\hat{n}}
\]
Thus the rate of rotation is largest in the direction of $\text{curl } \vec{F}$ and then $\omega = \frac{1}{2} |\text{curl } \vec{F}(p)|$.

**Note:** Vector fields with $\text{curl } \vec{F} = 0$ are called **irrotational**. Oddly, they can still rotate; experimentally, a draining tub is irrotational.

**Ex:** $\vec{F}(x, y, z) = \frac{1}{x^2 + y^2} (-y, x, 0)$

**Check:** $\text{curl } \vec{F} = 0$ except at $0$, where it doesn't make sense.

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**Conservative Vector Fields:** $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f : \mathbb{R}^n \to \mathbb{R}$

**Ex:** $\vec{F} = (x, y)$ conservative since $\vec{F} = \nabla (\frac{1}{2}(x^2 + y^2))$

$\vec{F} = (-y, x)$ not since $\frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y}$

$\begin{bmatrix} P \\ Q \end{bmatrix}$
Thm A: $\vec{F}$ on a connected set $D$ in $\mathbb{R}^n$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve $C$.

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$

Thm B: If $D$ in $\mathbb{R}^2$ is simply connected (no holes), then $\vec{F} = (P, Q)$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Missing Link: If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for each closed curve $C$.

Reason: As $D$ has no holes, the curve $C$ is the boundary of some region $R$.

Then

$\int_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$

$= \iint_R 0 \, dA = 0.$
Next time: What is Theorem B for \( \mathbb{R}^3 \)?

A start: Suppose \( \vec{F} = \nabla f = (f_x, f_y, f_z) \)

\[
\begin{align*}
curl \vec{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left( \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \\
&= 0.
\end{align*}
\]

Q: Is this enough?